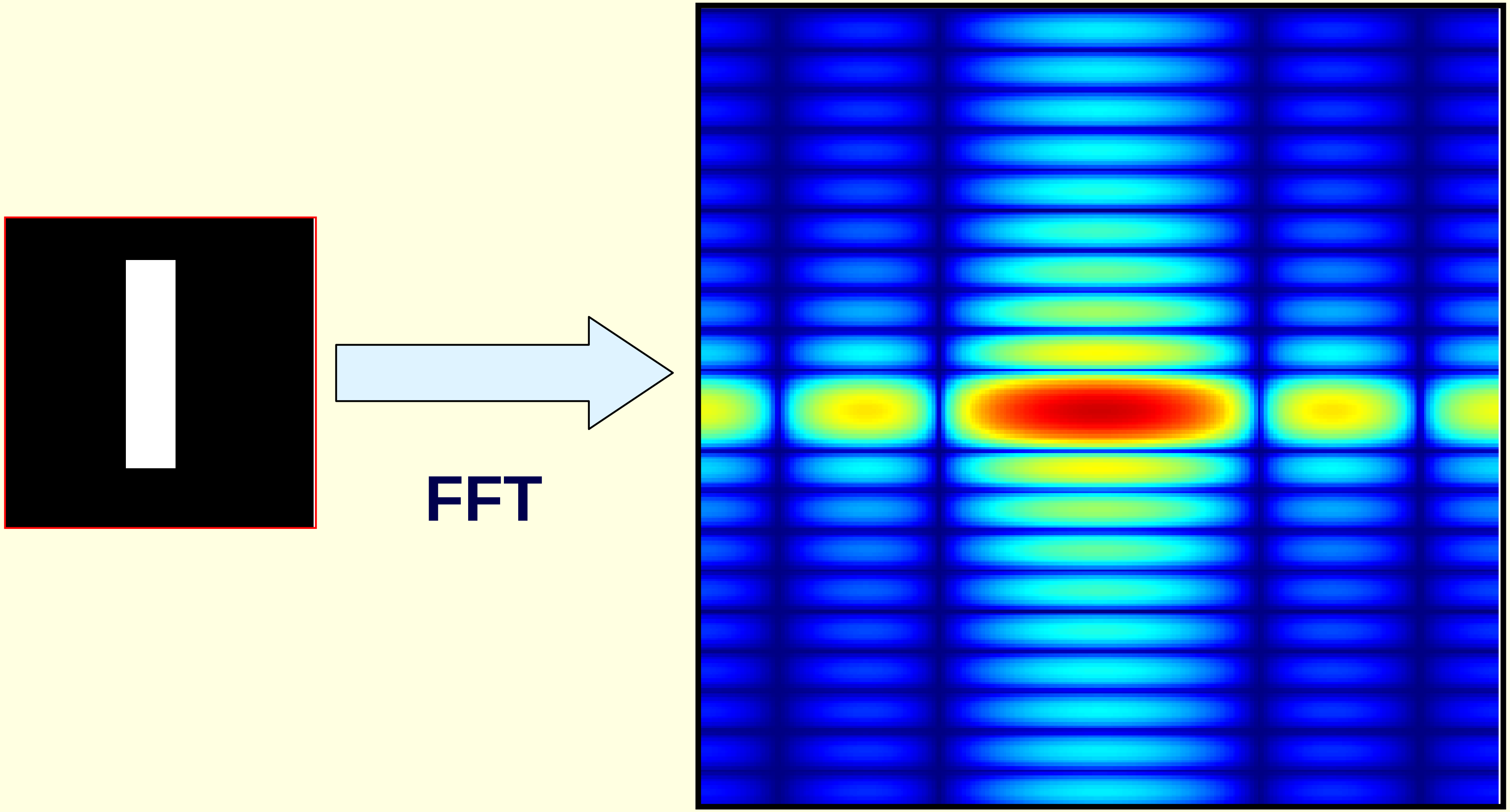
Fourier transform of images



Fourier transform

Joseph Fourier has put forward an idea of representing signals by a series of harmonic functions

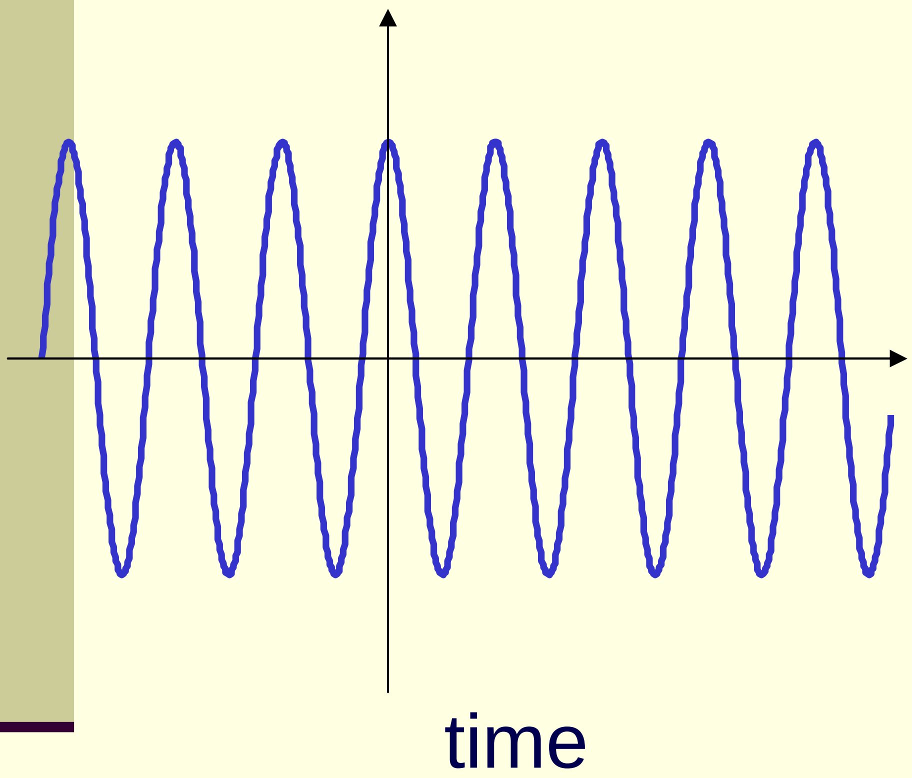
$$f(x) = \int_{-\infty}^{\infty} \underbrace{F(u)}\, e^{j2\pi ux} du$$ **inverse**

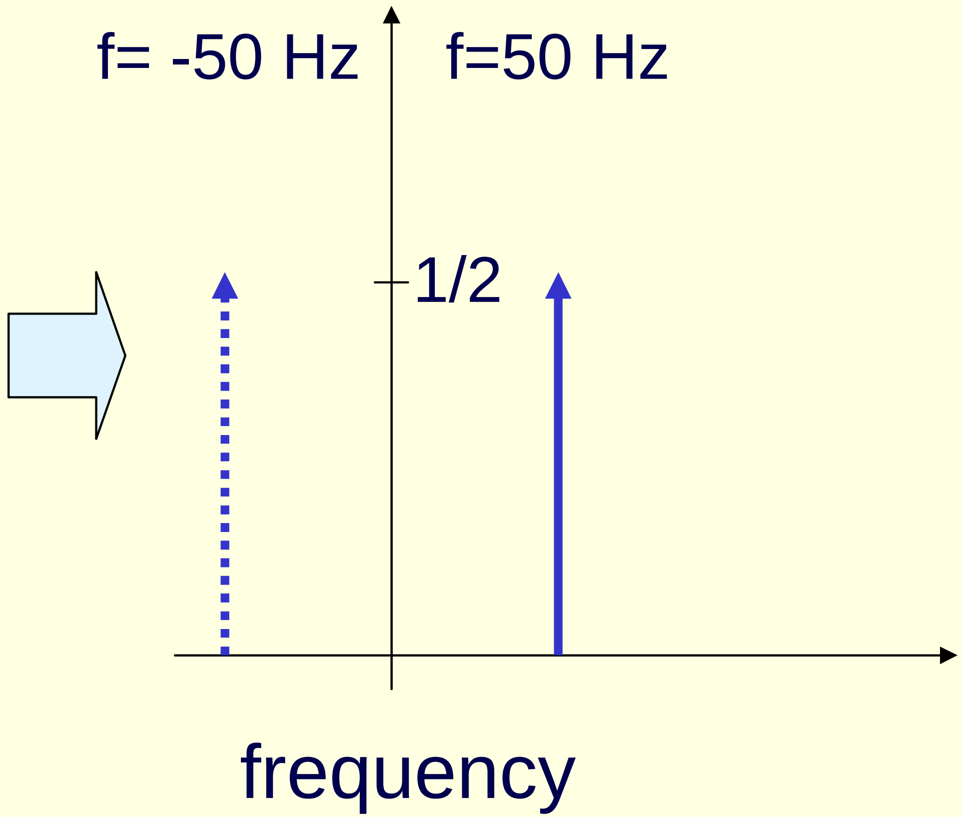
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$ **forward**

Fourier coefficients

Joseph Fourier
(1768-1830)

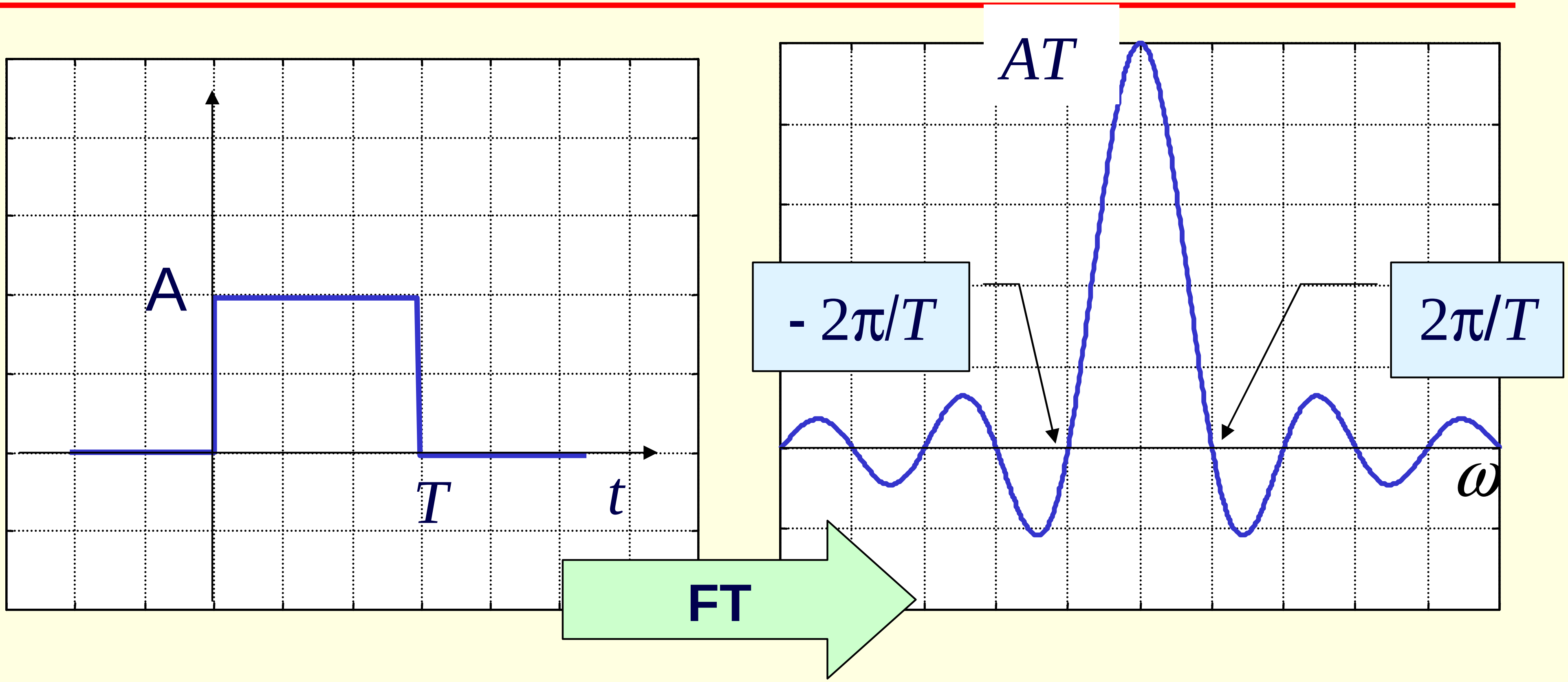
Fourier transform - example





$$X(j\omega) = \int_{-\infty}^{+\infty} \cos(\omega_0 t) e^{-j\omega t}\, dt = \frac{1}{2}\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

Fourier transform - example

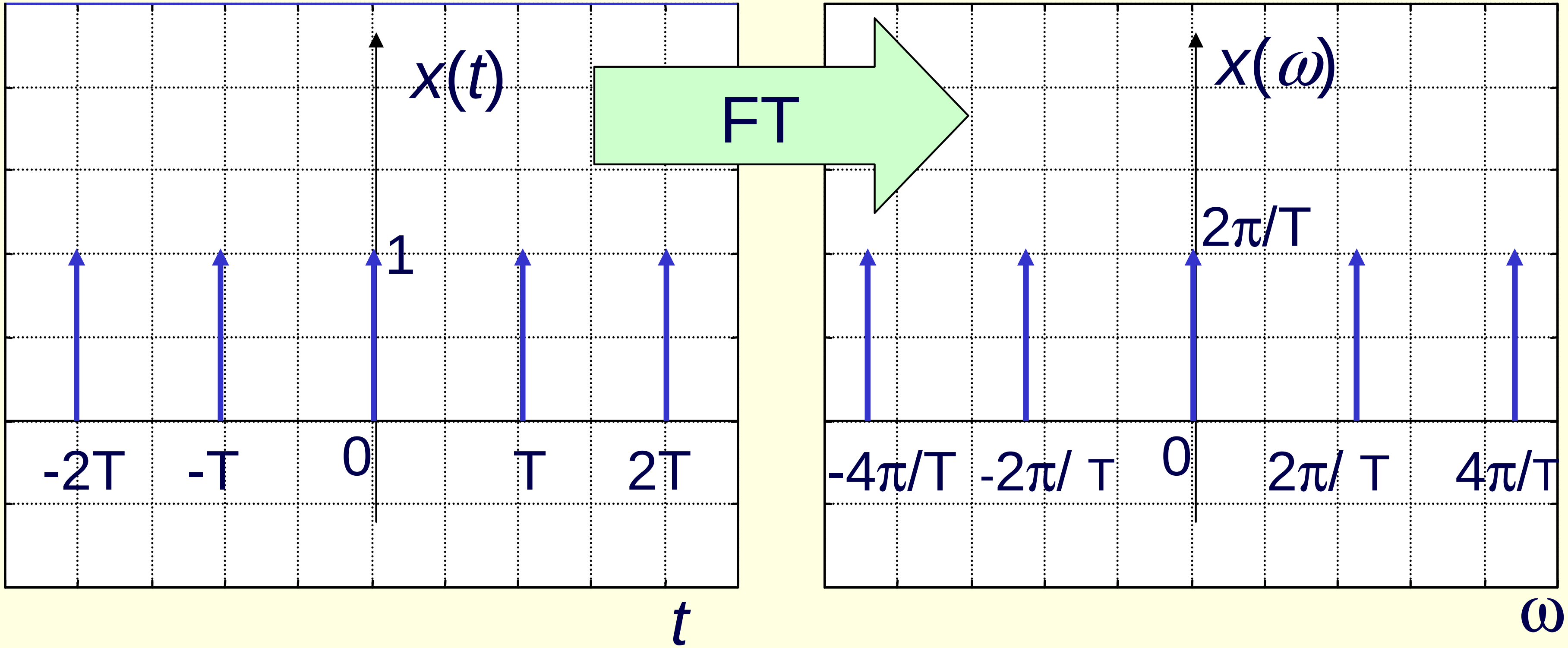


$$f(t)=\begin{cases}1, & t<T\\ 0, & t>T\end{cases}$$

$$\boxed{F(\omega)=A\int_0^T e^{-j\omega t}\,dt=\frac{2A}{\omega}\sin\left(\frac{\omega T}{2}\right)e^{-j\frac{\omega T}{2}}}$$

$$|F(\omega)|=?$$

Fourier transform - example



A series of Dirac pulses

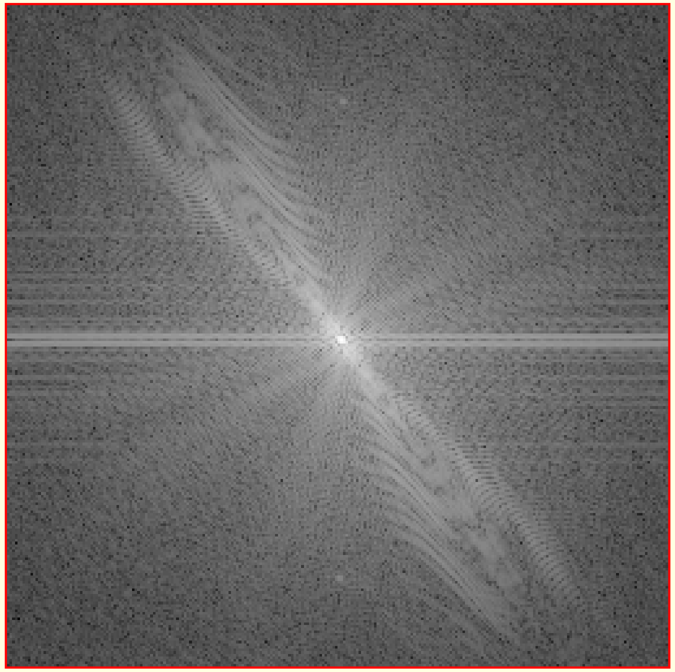
$$\delta_T(t) = \sum_{-\infty}^{+\infty} \delta(t - kT) \quad \leftrightarrow \quad \omega_s \sum_{-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

$$\omega_s = \frac{2\pi}{T}$$

Fourier transform of images

Monochrome image

Fourier spectrum



Why do we convert images (signals) to spectrum domain?

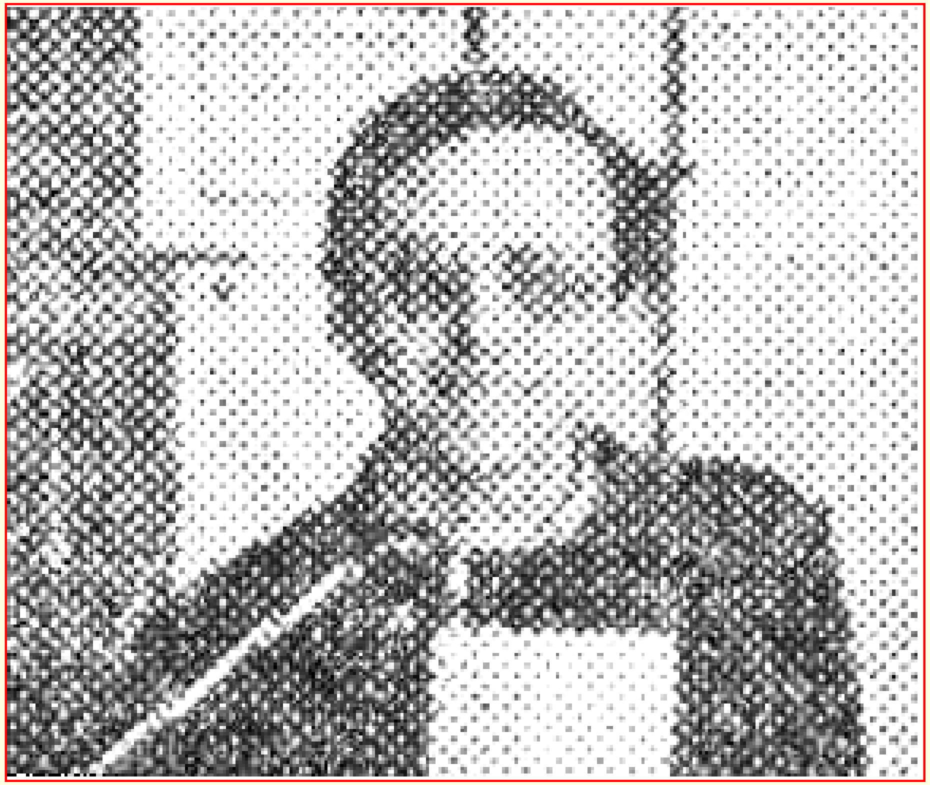
Fourier transform of images

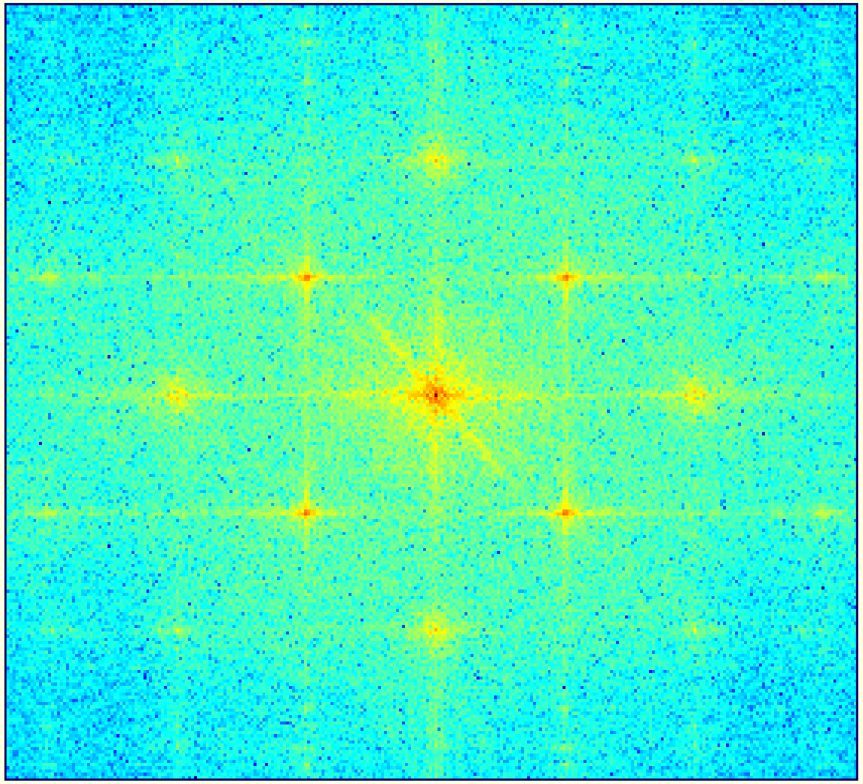
Why do we convert images to spectrum domain?

1. For **exposing image features** not visible in spatial domain, eg. periodic interferences
2. For achieving more compact image representation (coding), eg. **JPEG, JPEG2000**
3. For **designing digital filters**
4. For fast processing of images, eg. **digital filtering of images** in spectrum domain

Fourier transform of images

1. **Detection of image features**, eg. periodic interferences





Fourier transform of images

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$ **forward**

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$ **inverse**

Euler equations?

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\sin\omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

Amplitude and phase spectrum of the Fourier transform of images

$$F(u,v) = |F(u,v)|\, e^{-j\arg[F(u,v)]}$$

$$|F(u,v)| = \sqrt{\mathrm{Re}(F(u,v))^2 + \mathrm{Im}(F(u,v))^2}$$

$$\arg(F(u,v)) = \arctan \frac{\mathrm{Im}(F(u,v))}{\mathrm{Re}(F(u,v))}$$

The Discrete FT of images

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux+vy)/N}$$

$$dla \quad u,v=0,1,\ldots,N-1$$

$$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}F(u,v)e^{+j2\pi(ux+vy)/N}$$

$$dla \quad x,y=0,1,\ldots,N-1$$

Number of computations for 512x512 image?

1D computational example

$$f(x) = [1\ 3\ 4\ 4]$$

$$N = 4$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$F(u) = \frac{1}{N}\sum_{x=0}^{x=3} f(x)e^{-j2\pi ux/N}$$

$$F(0) = \frac{1}{N}\sum_{x=0}^{N-1=3} f(x)e^{-j2\pi 0x/N} = \frac{1}{4}\left[f(0) + f(1) + f(2) + f(3)\right] =$$

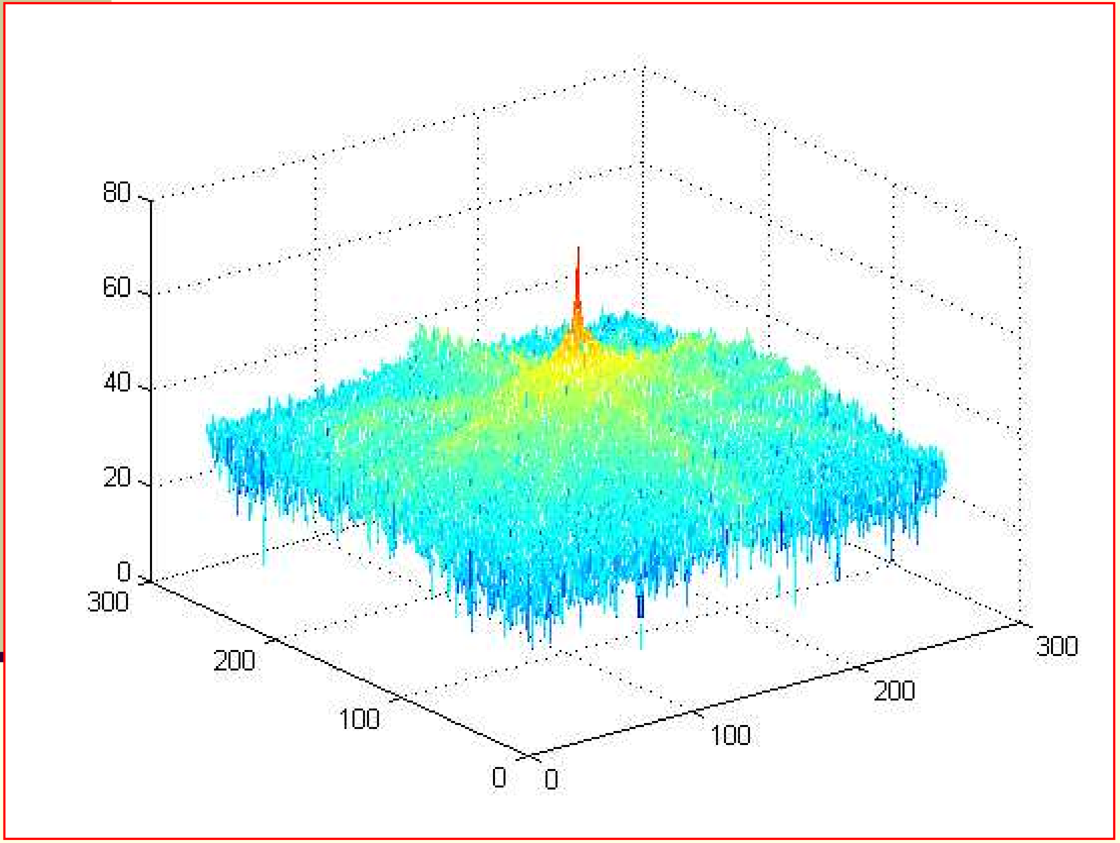
$$= \frac{1}{4}\left[1 + 3 + 4 + 4\right] = 3$$

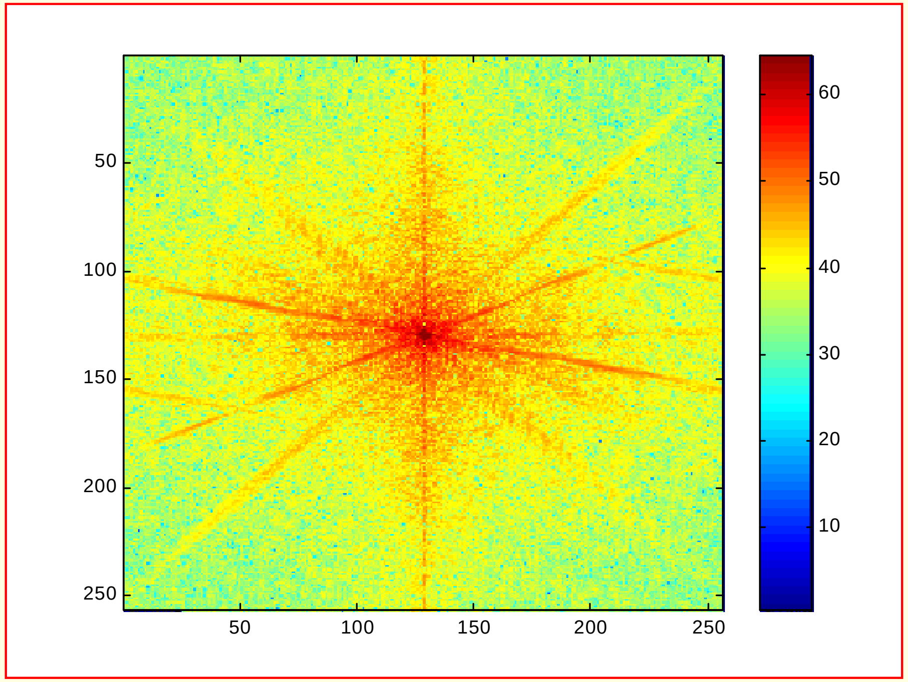
$$F(1) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{4}(-3 + j)$$

$$F(2) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = -\frac{1}{4}(2)$$

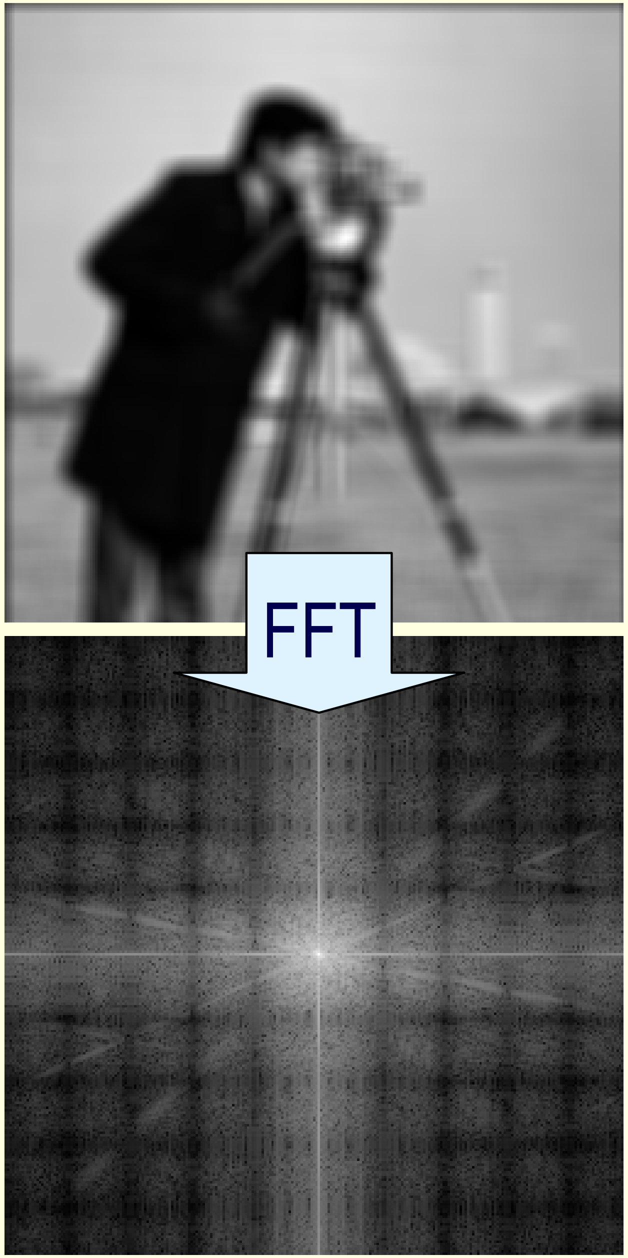
$$F(3) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = -\frac{1}{4}(3 + j)$$

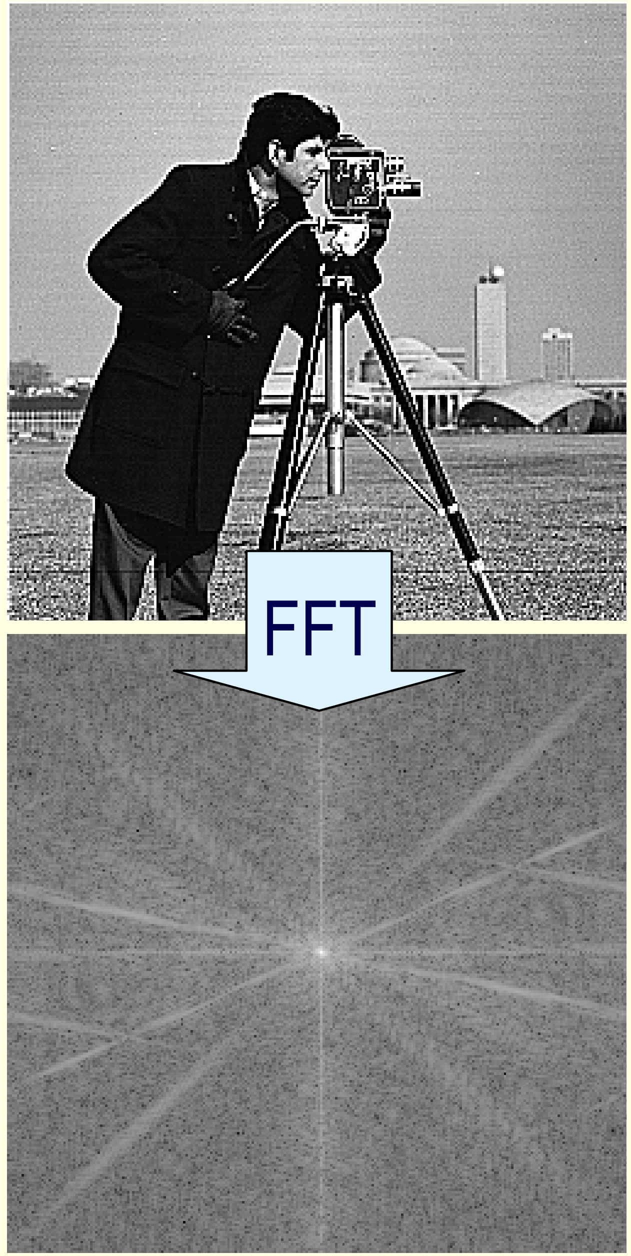
Fourier amplitude spectrum



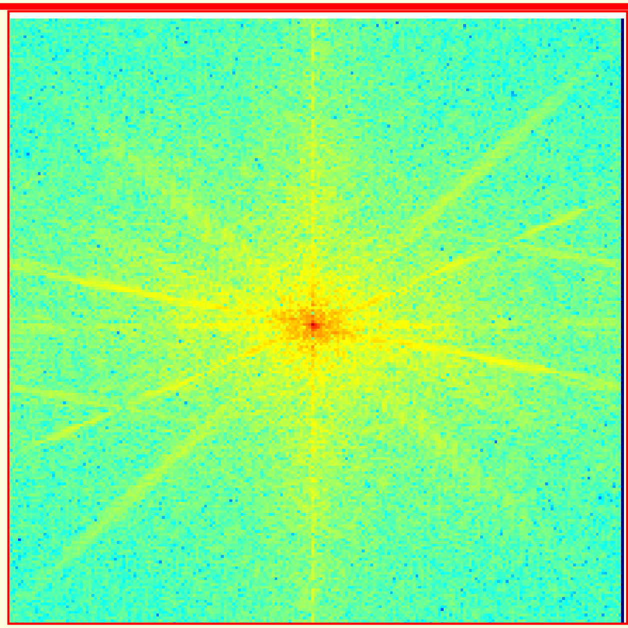


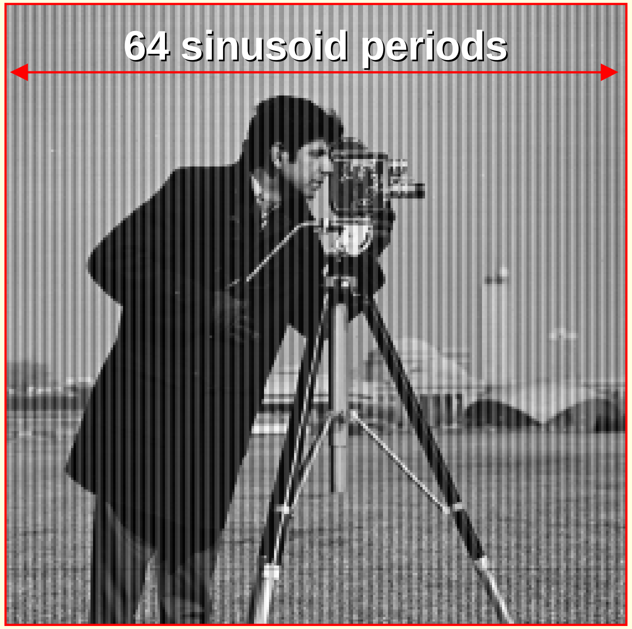
Fourier amplitude spectrum

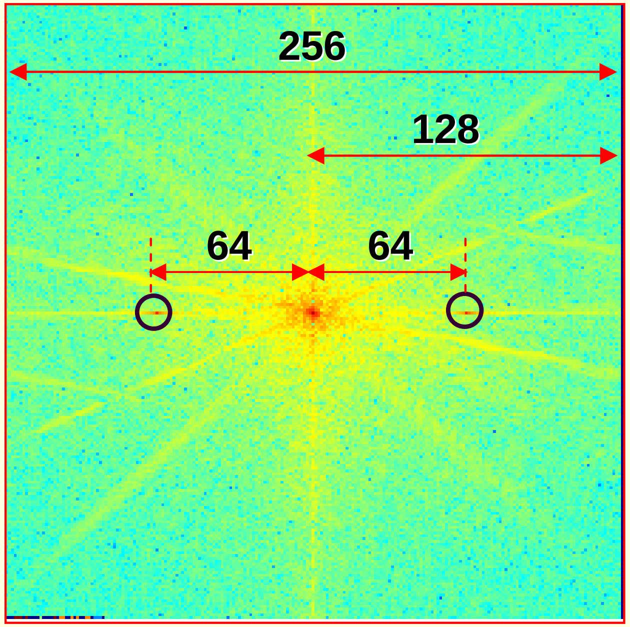




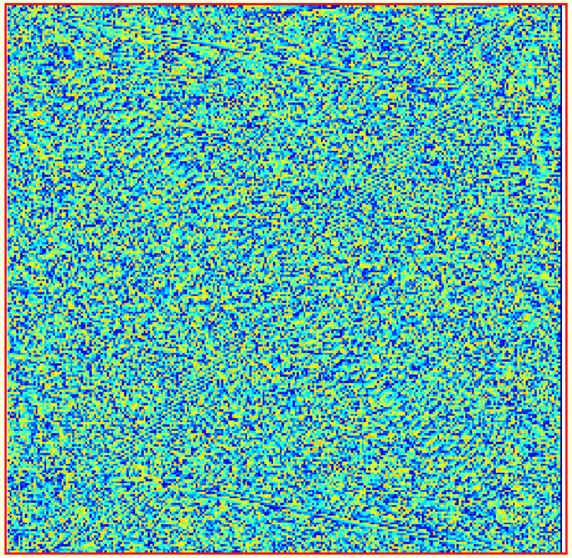
Detection of periodic distortions





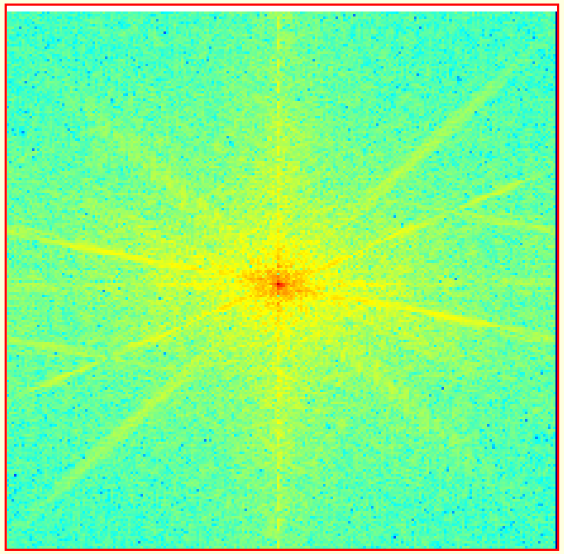


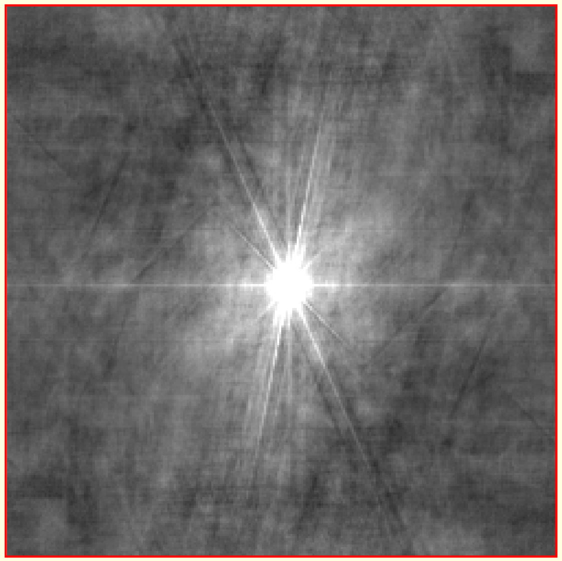
Fourier phase spectrum of an image



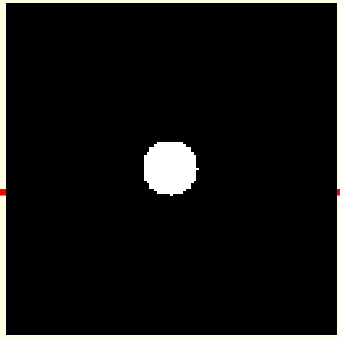
$\arg\left[F(u,v)\right]$

$\left|F(u,v)\right|$



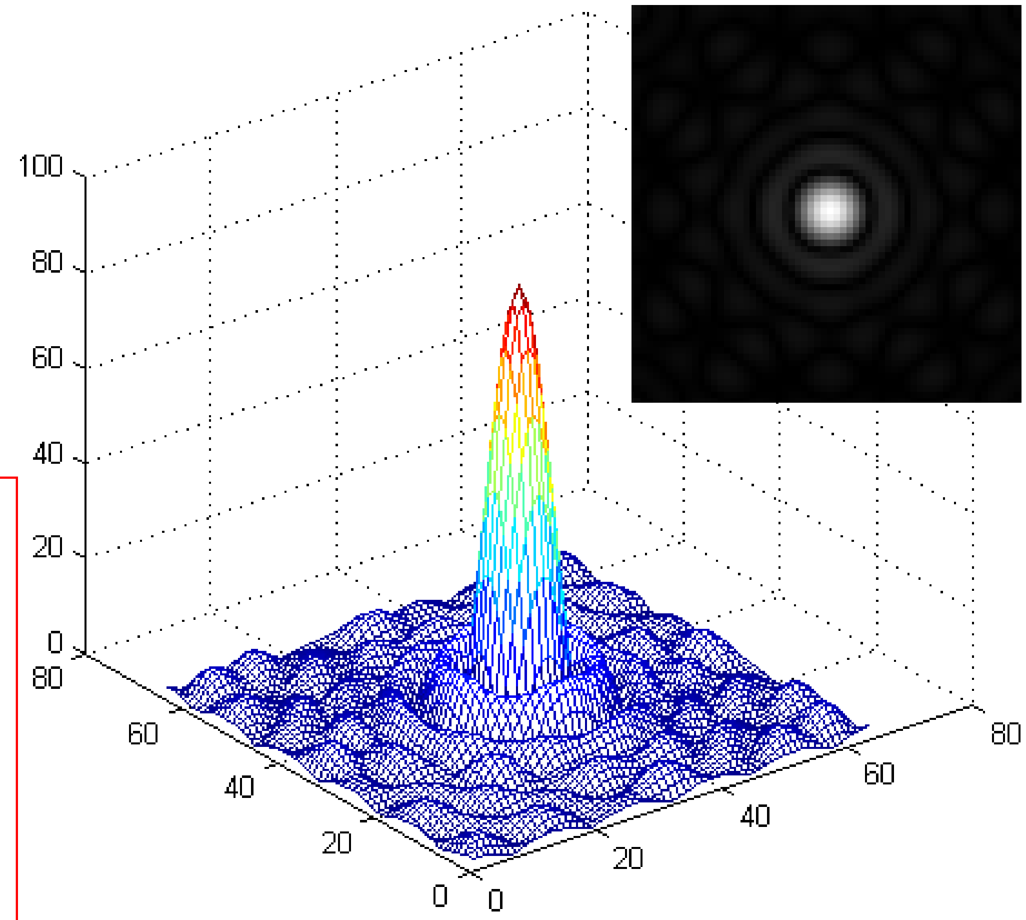


$\Im^{-1}\left\{\left|F(u,v)\right|\right\}$

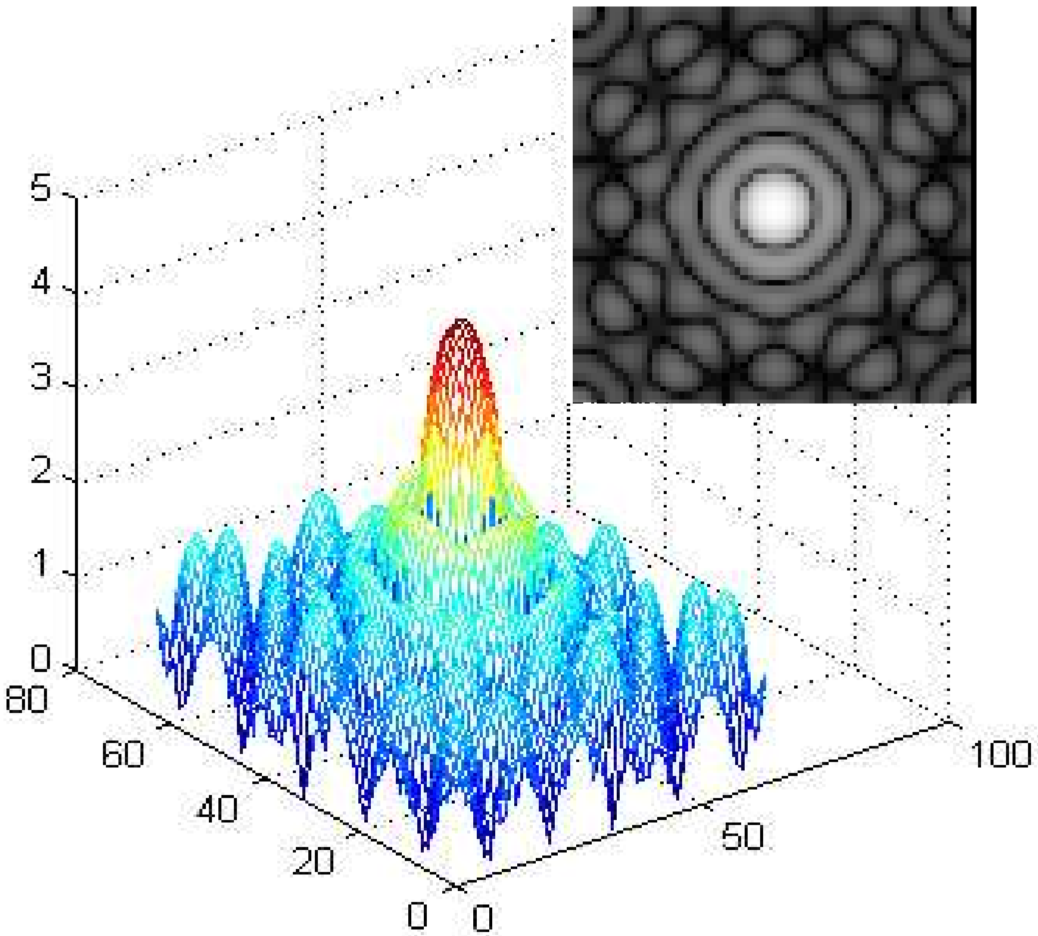


$f(x,y)$

(64x64)



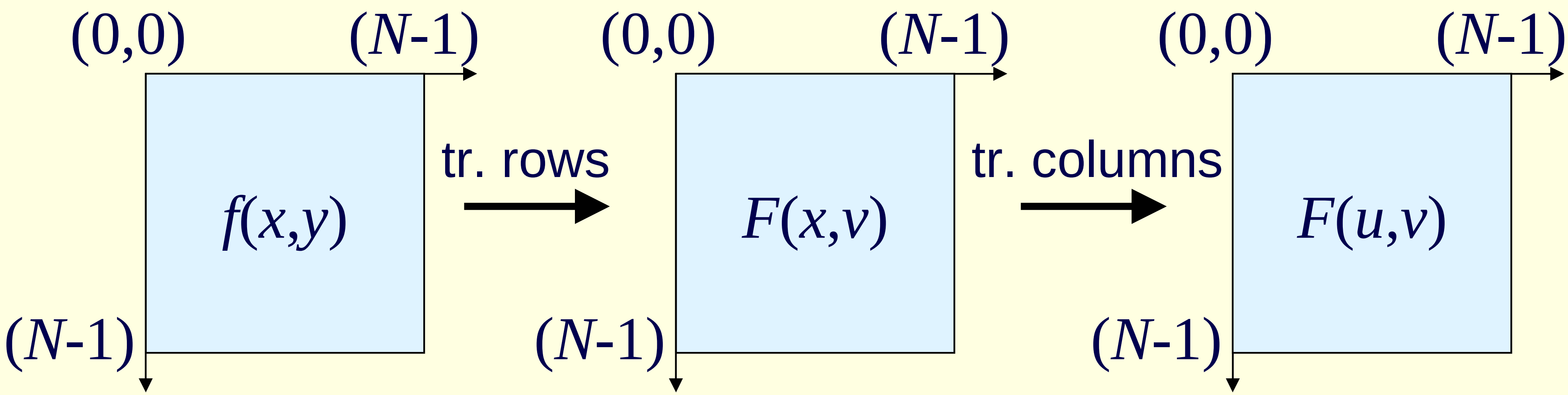
$|F(u,v)|$



$log(1+|F(u,v)|)$

Properties of the two-dimensional Fourier transform

Separability:



Computation of the 2-D Fourier transform as a series of 1-D transforms

Separability of the 2-D Fourier transform

$$F(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux+vy)/N}$$

$$F(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} e^{-j2\pi ux/N} \overbrace{\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi vy/N}}^{F(x,v)}$$

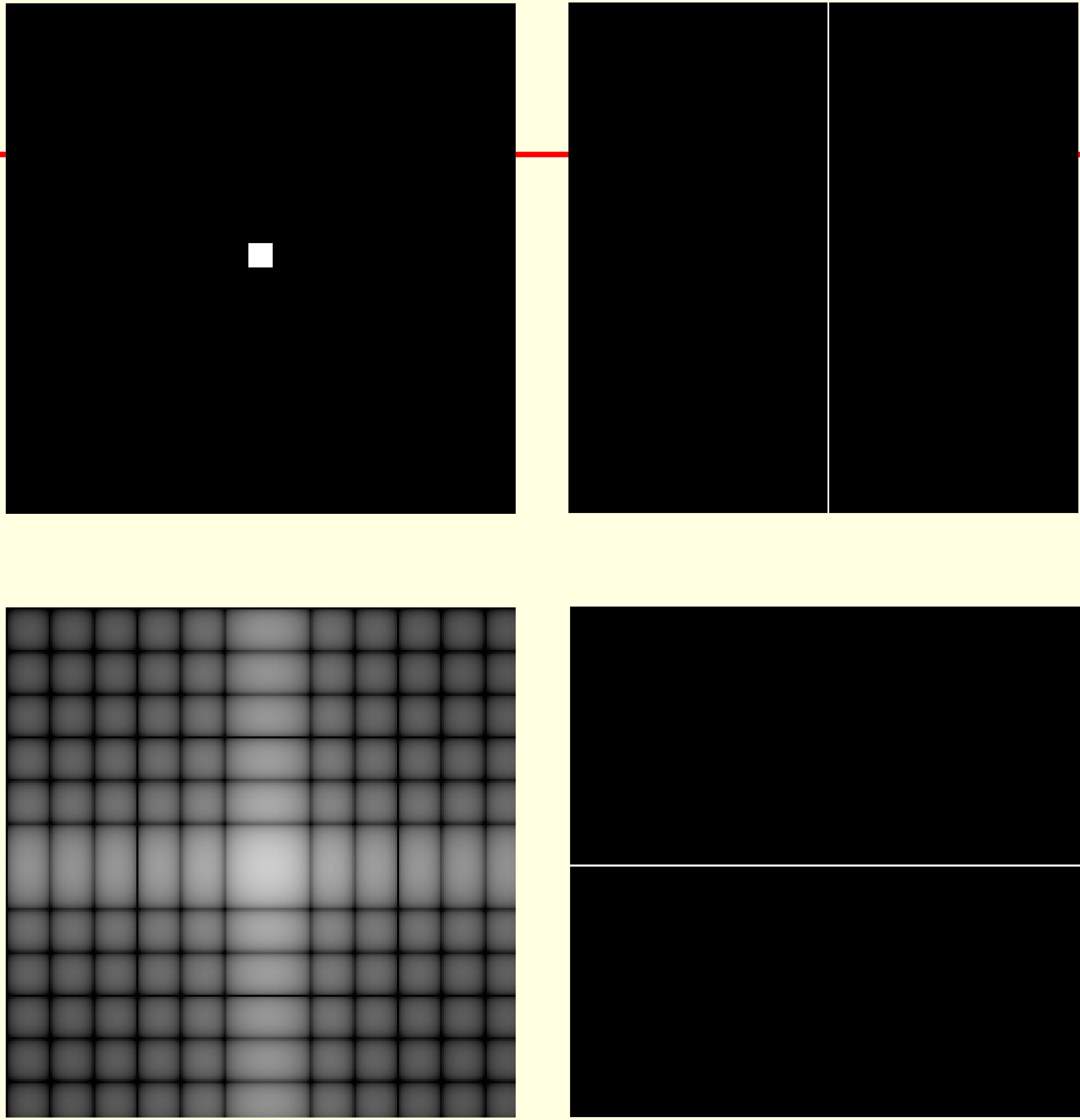
$$F(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} F(x,v) e^{-j2\pi ux/N}$$

Example

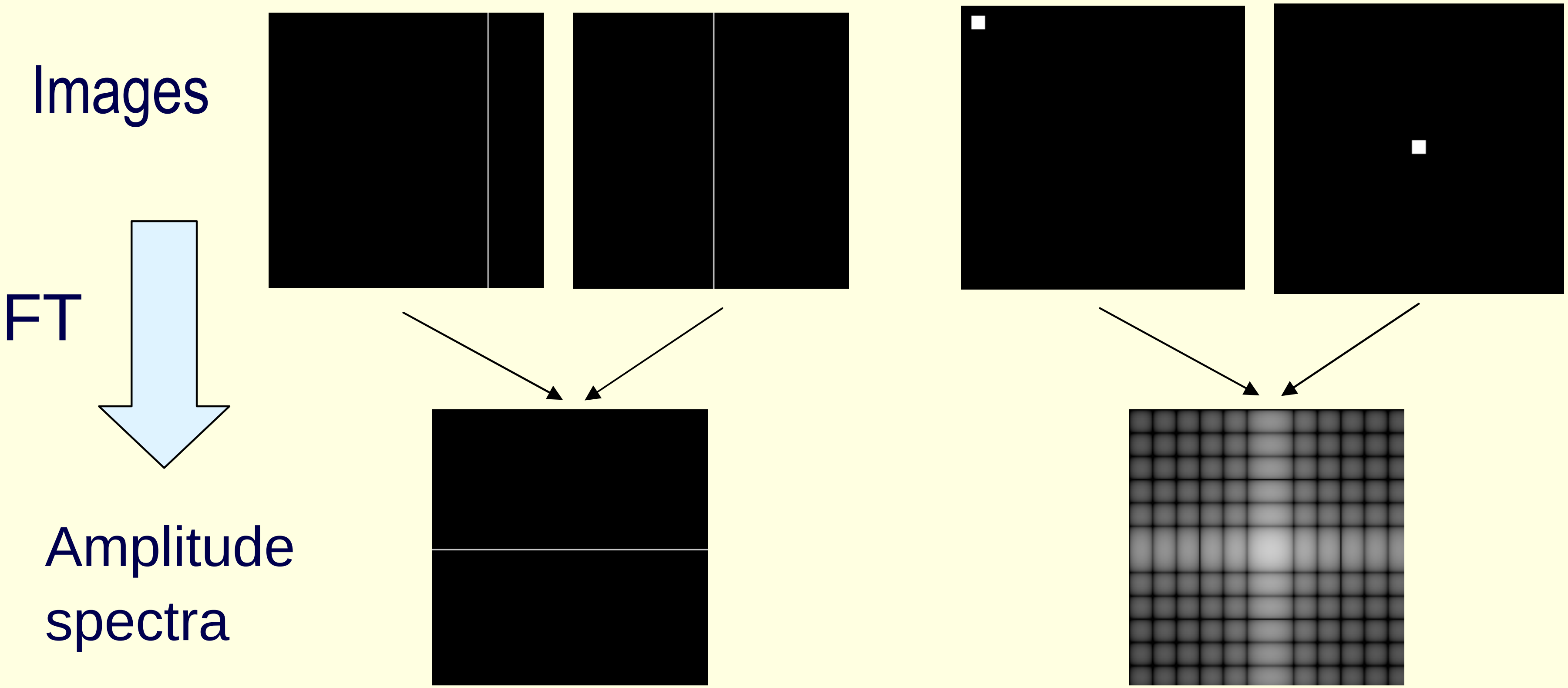
Images

FT

Amplitude spectra



Shift in the spatial domain



$$f\left(x-x_0, y-y_0\right) \Leftrightarrow F\left(u,v\right) exp\left[-\frac{j2\pi\left(ux_0+vy_0\right)}{N}\right]$$

Properties of the two-dimensional Fourier transform

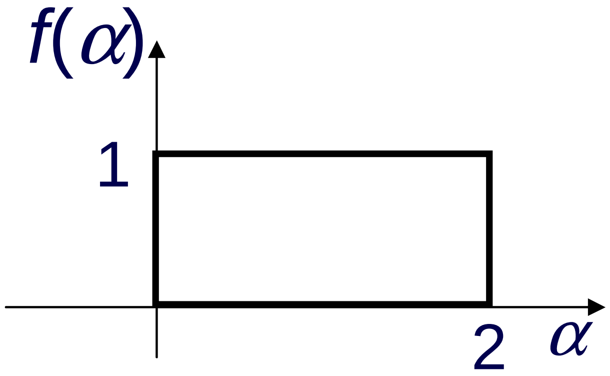
Convolution:

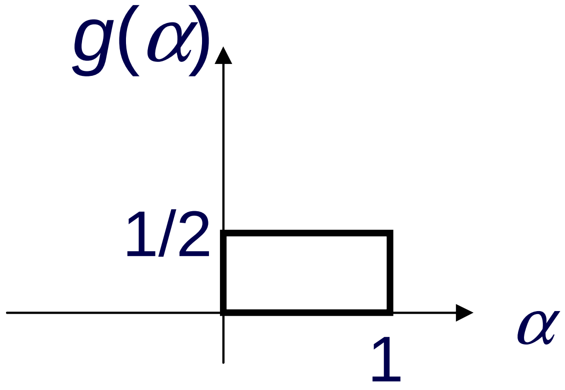
$$\mathcal{F}\{f(x,y)\, g(x,y)\} = F(u,v) * G(u,v)$$

$$\mathcal{F}\{f(x,y) * g(x,y)\} = F(u,v)\, G(u,v)$$

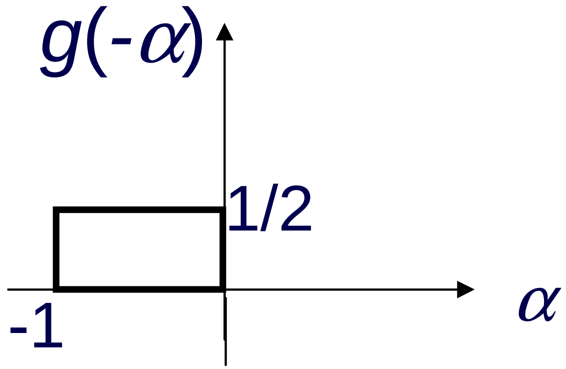
This property is useful in designing digital image filters.

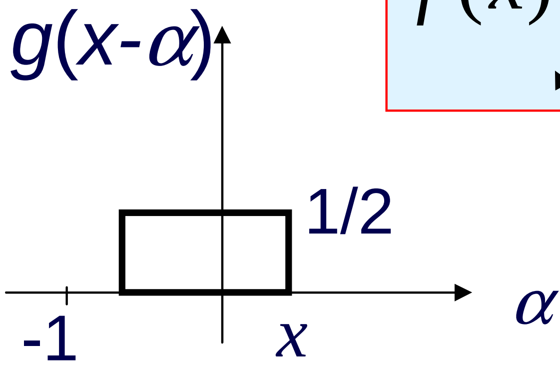
1-D convolution example

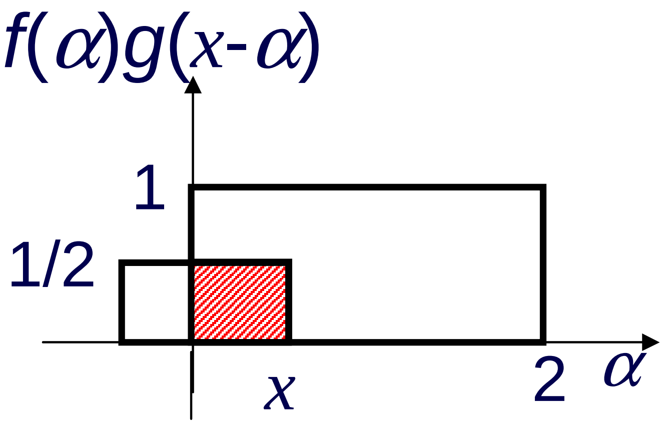


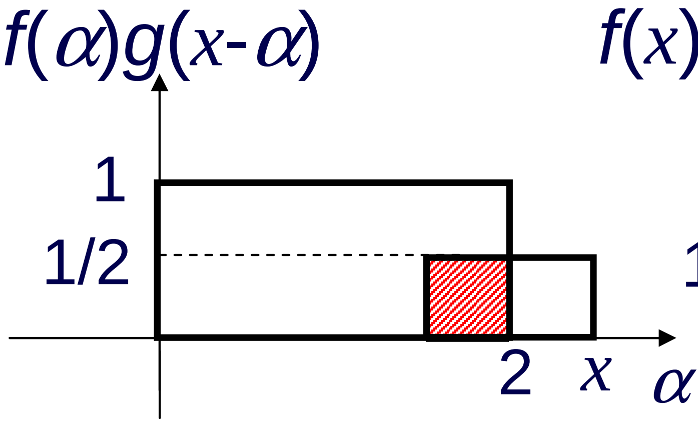


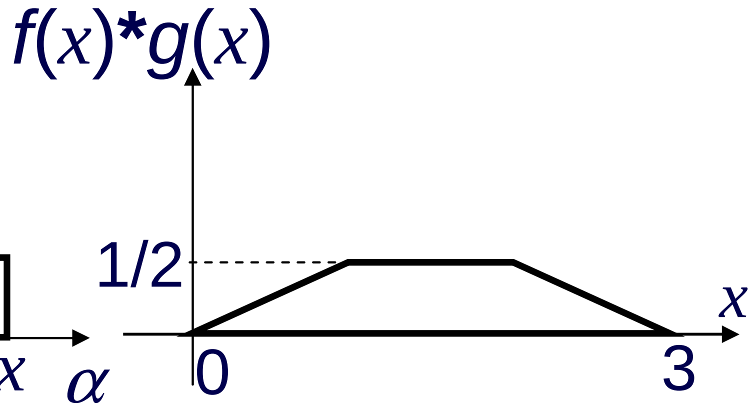
$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) d\alpha$$











2-D convolution of discrete functions

f(*i,j*), *g*(*i,j*) – dicretete 2-D functions of period *NxN*

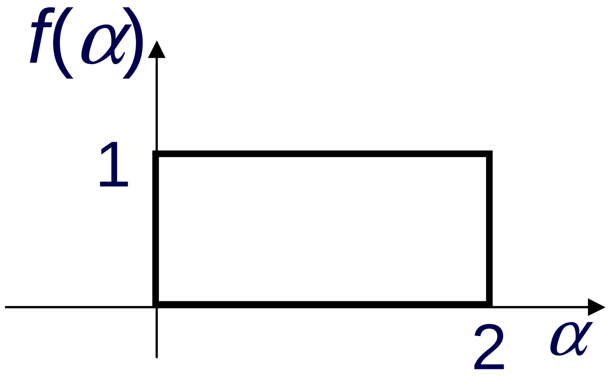
increase periods of *f*(*i,j*) and *g*(*i,j*) up to *M=2N-1*:

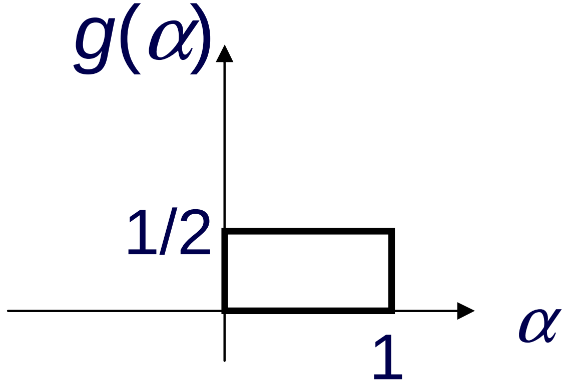
$$f_e(i,k)=\begin{cases} f(i,k) & 0\le i,k\le N-1 \\ 0 & N\le i,k\le M-1 \end{cases} \qquad g_e(i,k)=\begin{cases} g(i,k) & 0\le i,k\le N-1 \\ 0 & N\le i,k\le M-1 \end{cases}$$

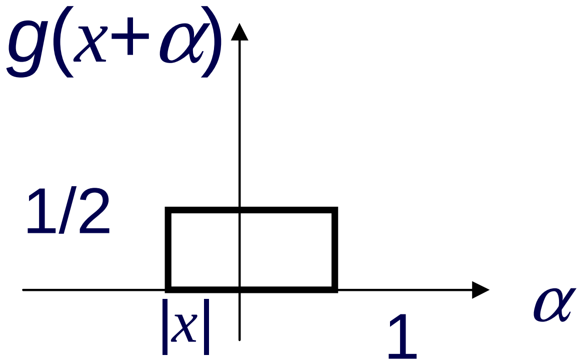
$$f_e(i,k)*g_e(i,k)=\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} f_e(m,n)\,g_e(i-m,k-n)$$

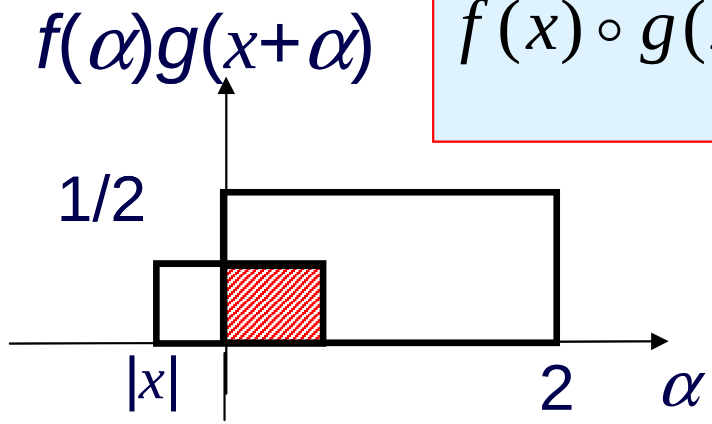
1-D correlation - example

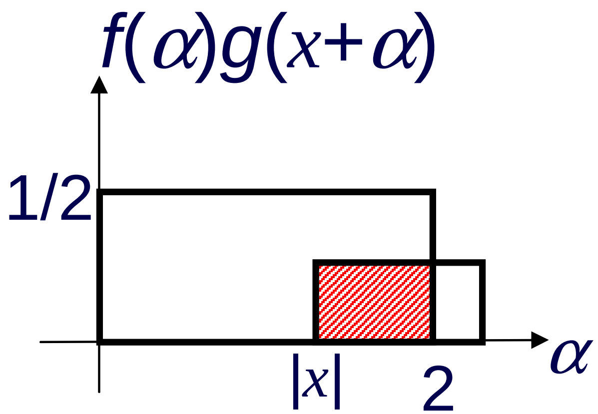
$$f(x) \circ g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x+\alpha) d\alpha$$

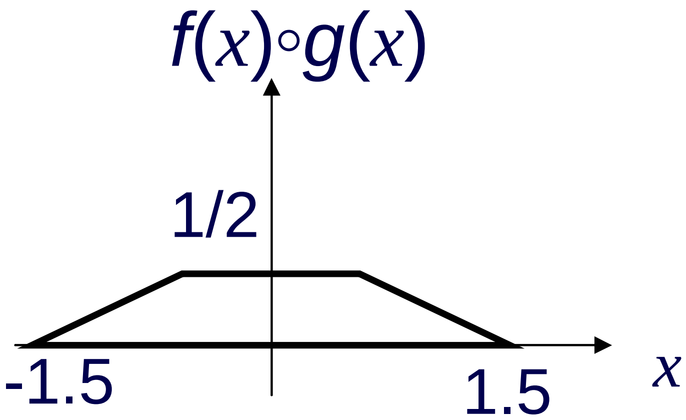












Correlation of 2-D discrete functions

f(*i,j*), *g*(*i,j*) – dicretete 2-D functions of period *NxN*

Increase the periods as for convolution:

$$f_e(i,k) \circ g_e(i,k) = \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} f_e(m,n) g_e(i+m,k+n)$$

Periodicity of the FT

$$F(u,v)=F(u+N,v)=F(u,v+N)=F(u+N,v+N)$$

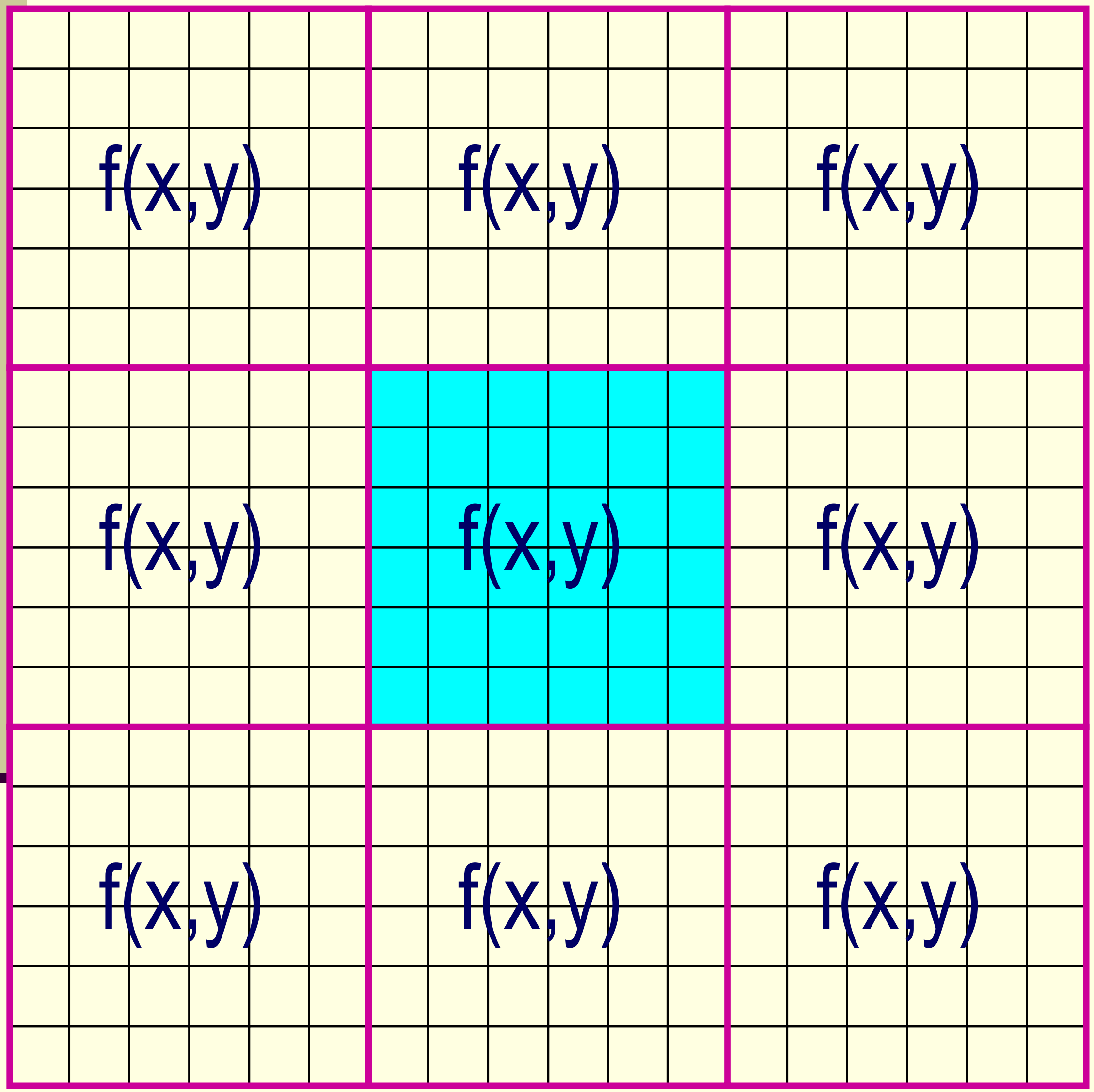
If f(x,y) is a real valued function then:

$$F(u,v) = F^*(-u,-v)$$

and:

$$|F(u,v)| = |F(-u,-v)|$$

Fourier transform of images



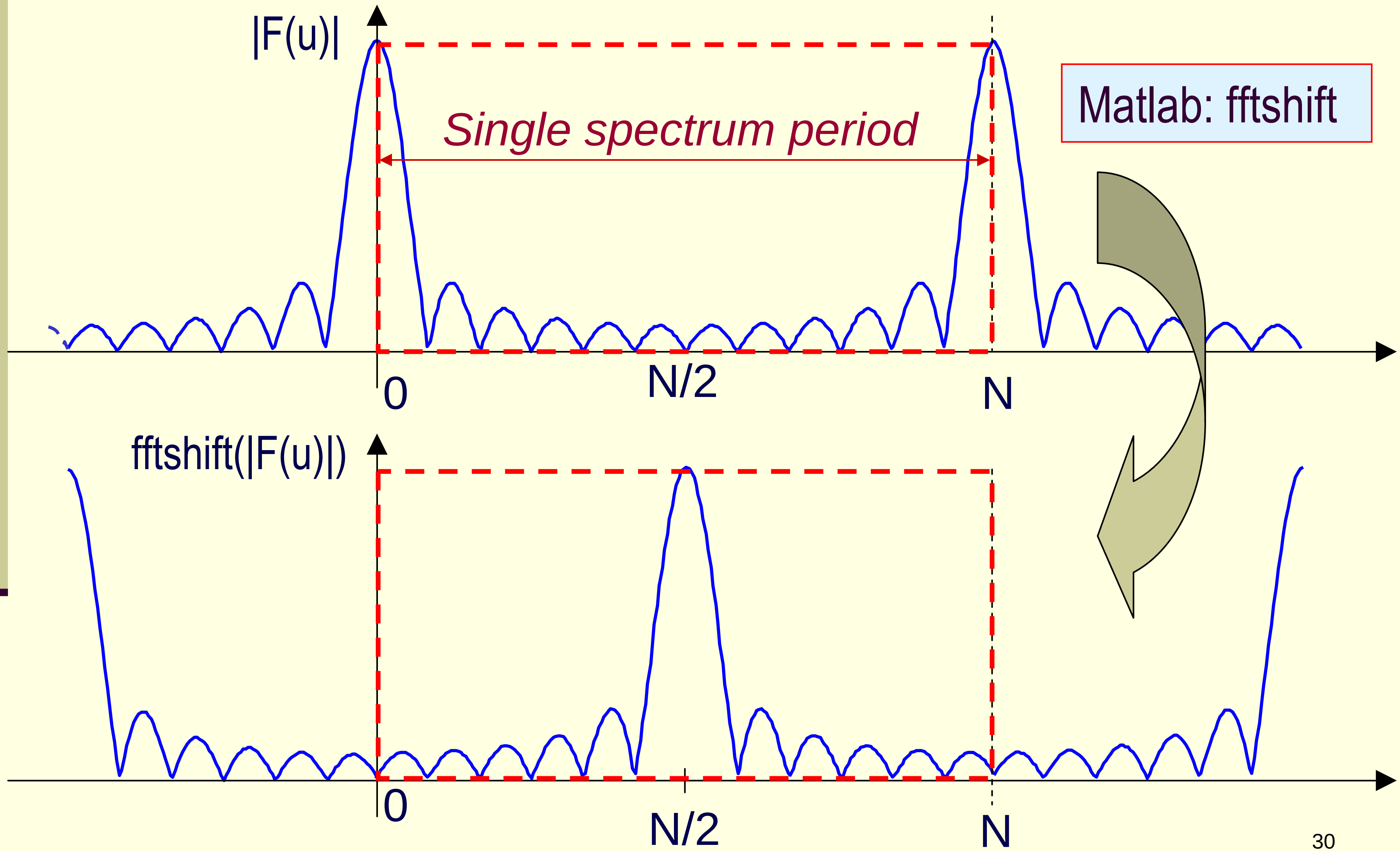
It is assumed the transformed image is a periodic function of period (N, N)

Translation in the spectral domain

$$\boxed{f(x,y)\exp\left[\frac{j2\pi(u_0x+v_0y)}{N}\right]\Leftrightarrow F(u-u_0,v-v_0)}$$

This Fourier property is known as the theorem of modulation.

Translation in the spectral domain



Translation in the spectral domain

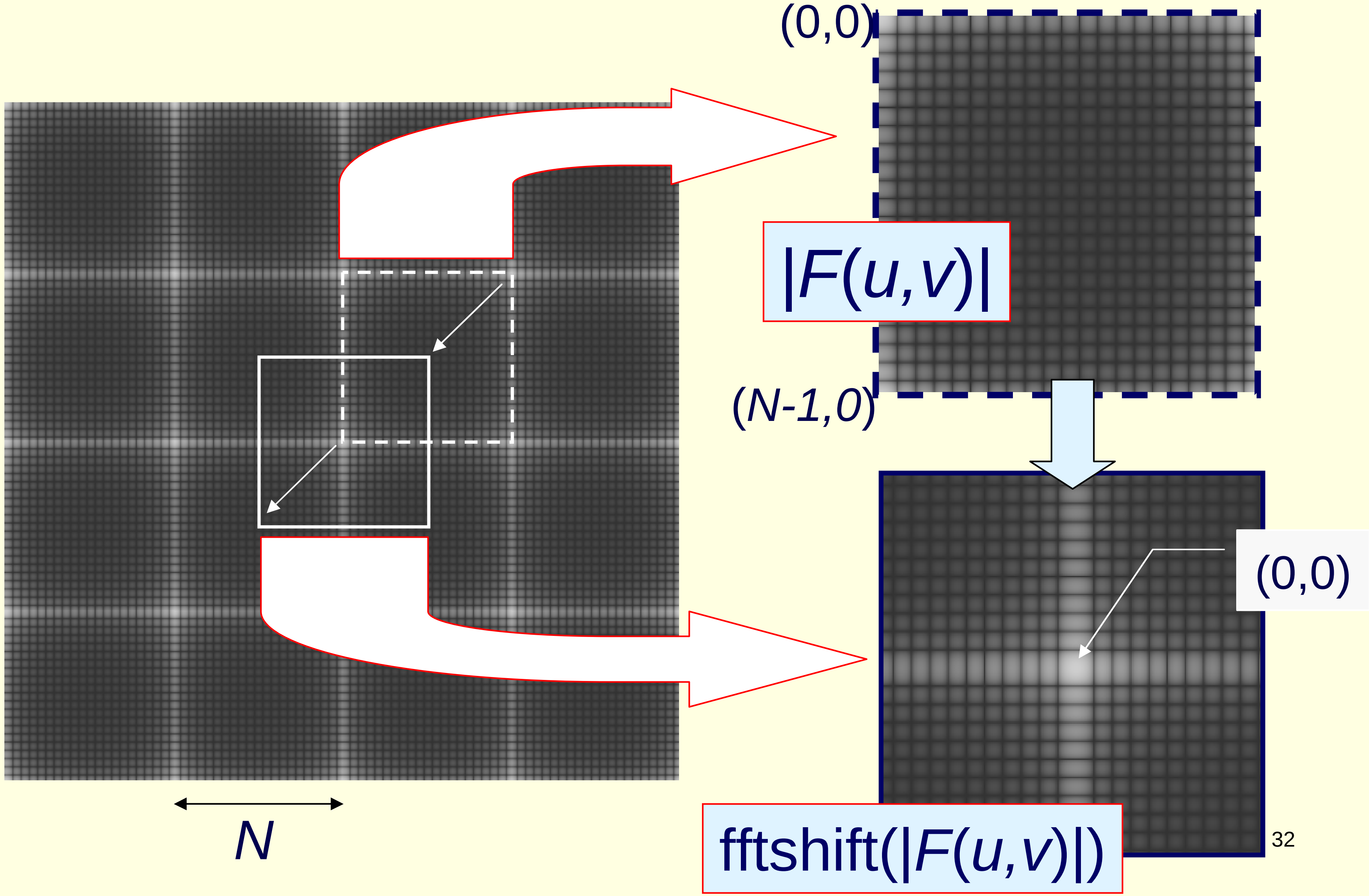
$$f(x,y)\exp\left[\frac{j2\pi(u_0x+v_0y)}{N}\right] \Leftrightarrow F(u-u_0,v-v_0)$$

$$for\ u_0=v_0=\frac{N}{2} \quad \Leftrightarrow F\left(u-\frac{N}{2},v-\frac{N}{2}\right)$$

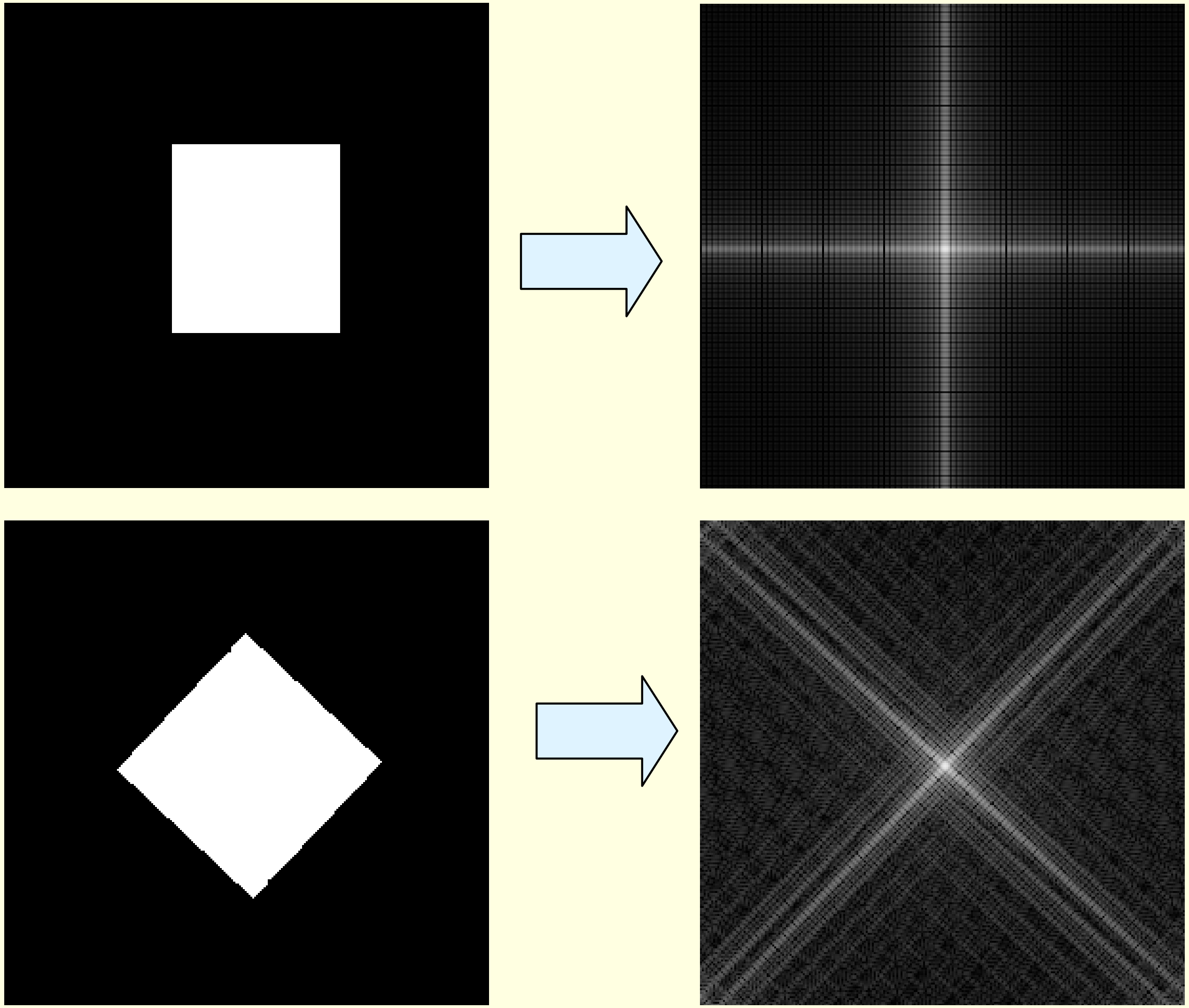
$$f(x,y)\exp\left[\frac{j2\pi(u_0x+v_0y)}{N}\right]=$$

$$=f(x,y)\exp[j\pi(x+y)]=f(x,y)(-1)^{x+y}$$

Translation in spectral domain



Rotation



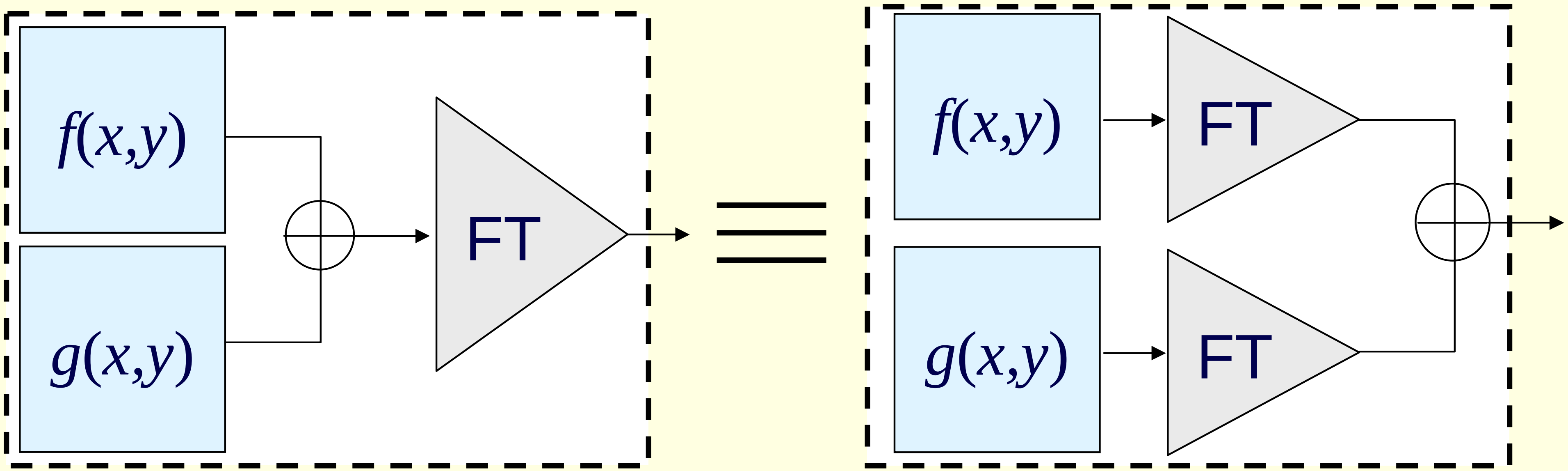
$\theta_0 = 0°$

$\theta_0 = 45°$

$$f(r,\ \theta + \theta_0) \Leftrightarrow F(\omega,\ \phi + \theta_0)$$

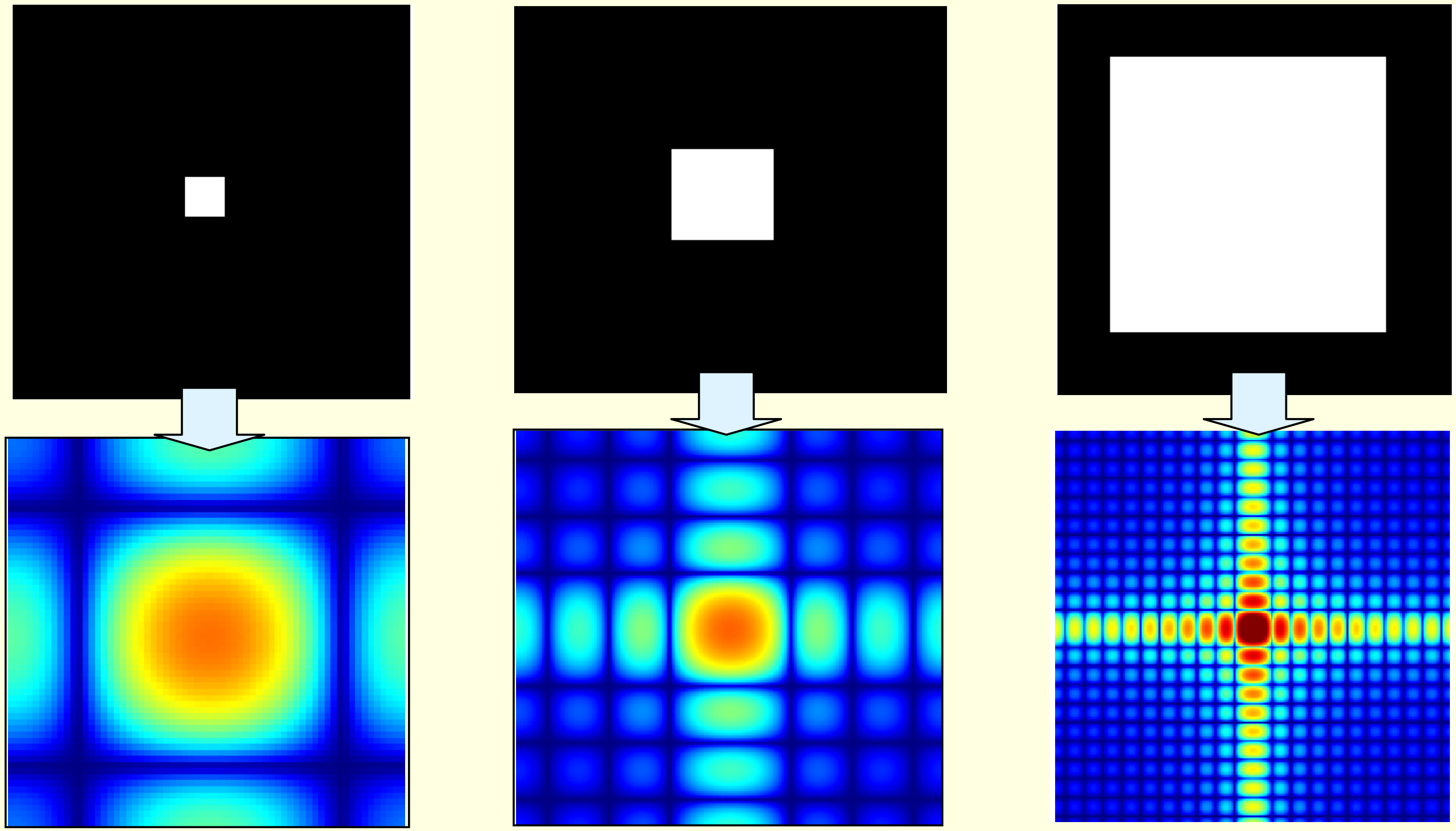
Linearity

$$\Im\{a\, f(x,y) + b\, g(x,y)\} = a\, F(u,v) + b\, G(u,v)$$



Scaling

$$\Im\{f(ax,by)\} = |ab|^{-1} F(u/a,\, v/b) \qquad a,b \in R$$

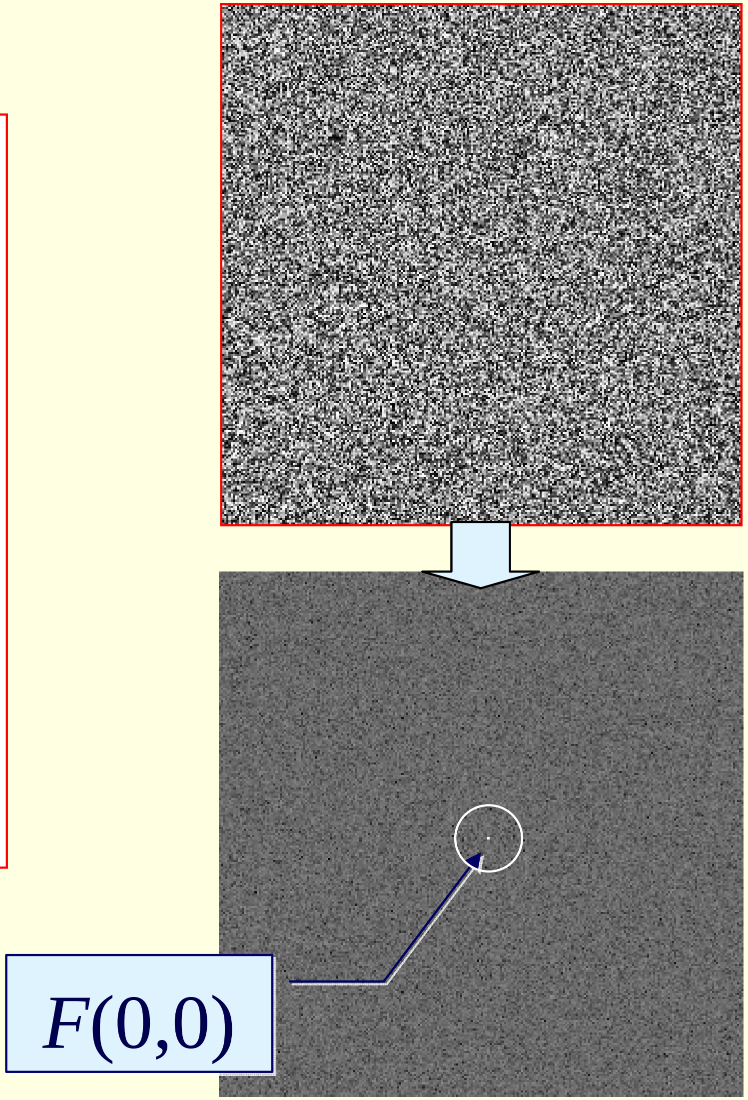


Average value

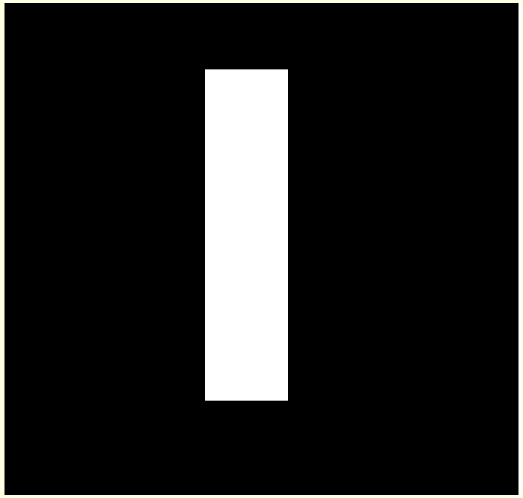
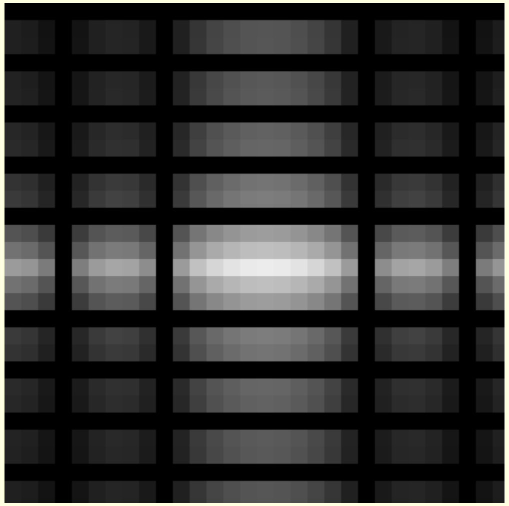
$$\bar{f}(x,y)=\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

$$F(0,0)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

$$\bar{f}(x,y)=\frac{1}{N}F(0,0)$$



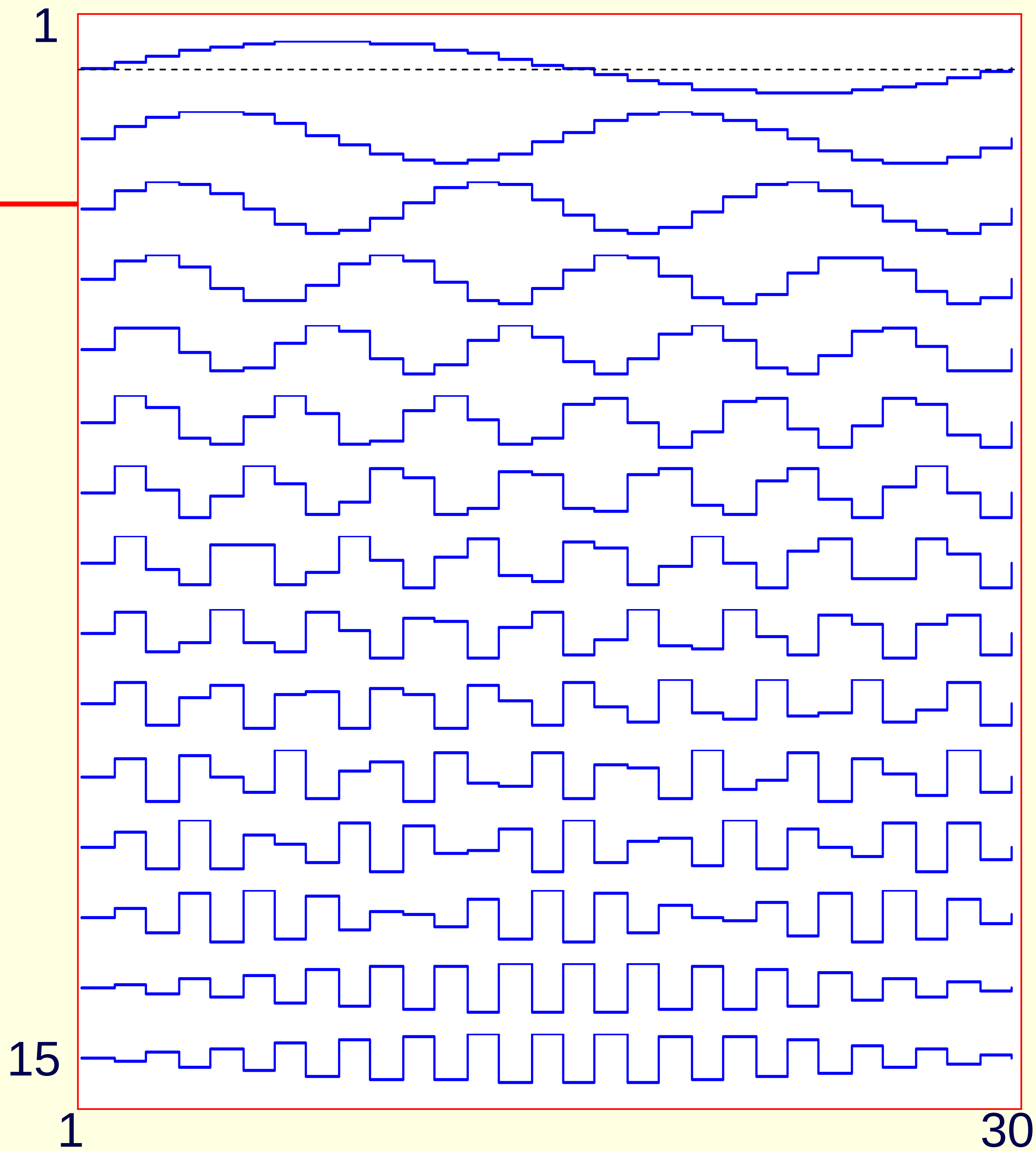
Fourier transform of an image - examples

f   |F1|

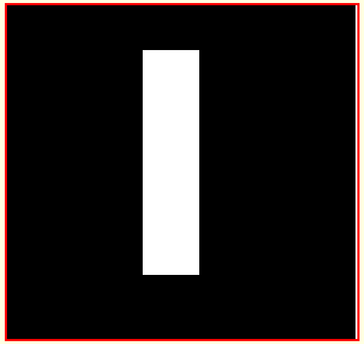
```
f=zeros(30,30);
f(5:24,13:17)=1;

imshow(f,'notruesize')
F=fft2(f);                  %compute 2-D Fourier transform
F1=log(abs(F)+1);           %amplitude spectrum
imshow(F1,[0 5],'notruesize');
```

Discrete Fourier Transform

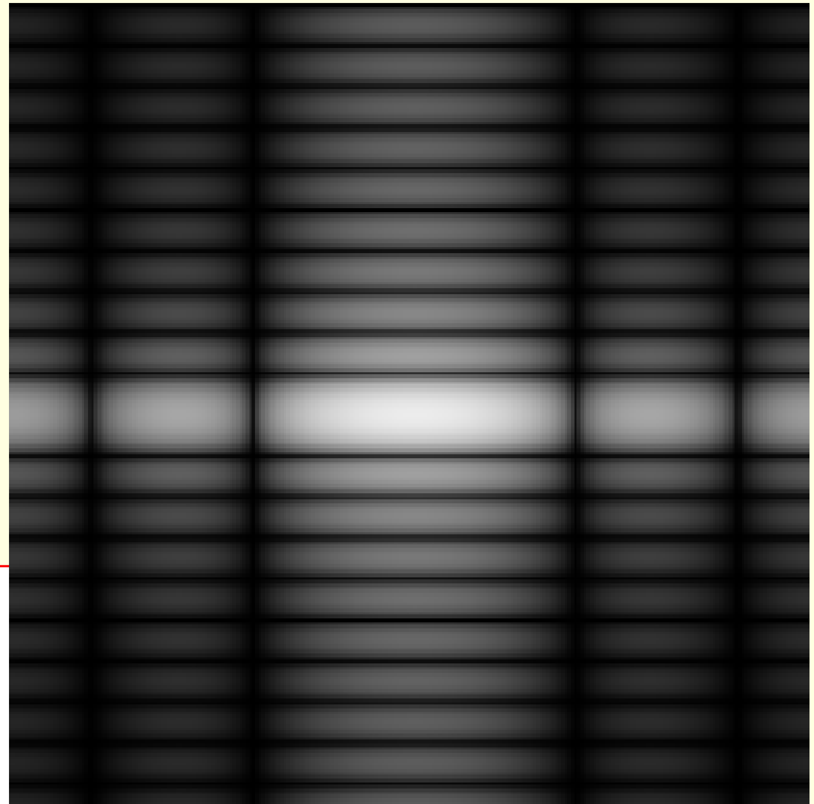
Basis functions for 30-point Fourier transform (sine component)



Fourier transform of an image - examples



f



|F2|

```
%better resolution
f=zeros(30,30);
f(5:24,13:17)=1;
F=fft2(f,256,256);
F2=log(abs(F)+1);
imshow(fftshift(F2),[0 5],'notruesize');
```

The Fast Fourier Transform, FFT (*succesive doubling method*)

If $N=2^n$, then $N=2*M$ and one can show that:

$$F_{even}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x)W_M^{ux}, \quad F_{odd}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)W_M^{ux}$$

$$F(u) = \frac{1}{2}[F_{even}(u) + F_{odd}(u)W_{2M}^{u}], \qquad u = 0,1,\ldots,M-1$$

$$F(u+M) = \frac{1}{2}[F_{even}(u) - F_{odd}(u)W_{2M}^{u}], \qquad u = 0,1,\ldots,M-1$$

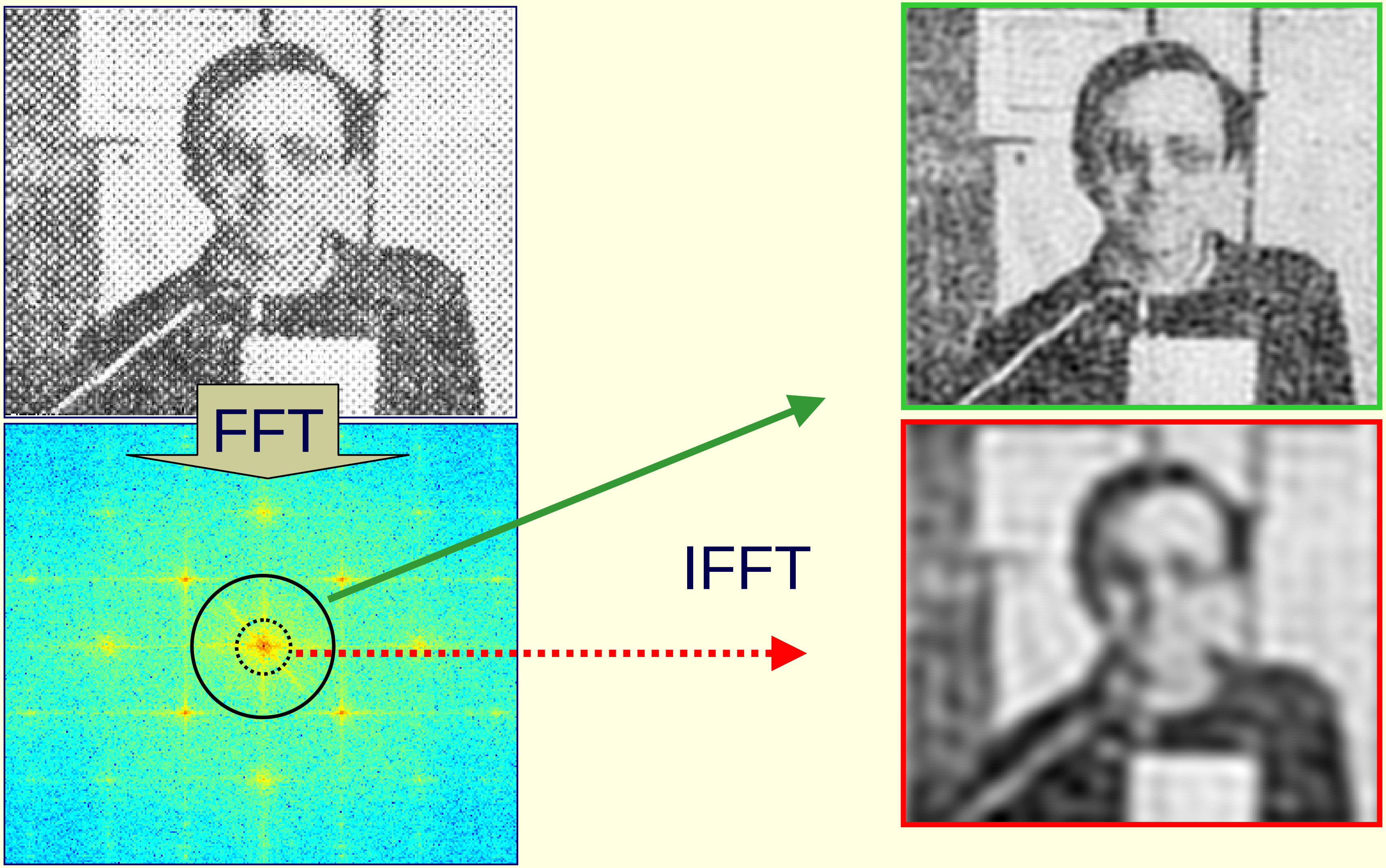
$$\boxed{W_M = e^{-j2\pi/M}}$$

Comparison of TF and FFT

N	N^2 (FT)	NlogN (FFT)	Advantage N/logN
16	256	64	4
256	65535	2048	32
512	262144	4608	64
2048	~4e6	22528	186

2-D Fourier transform

Interactive noise reduction in Fourier spectrum

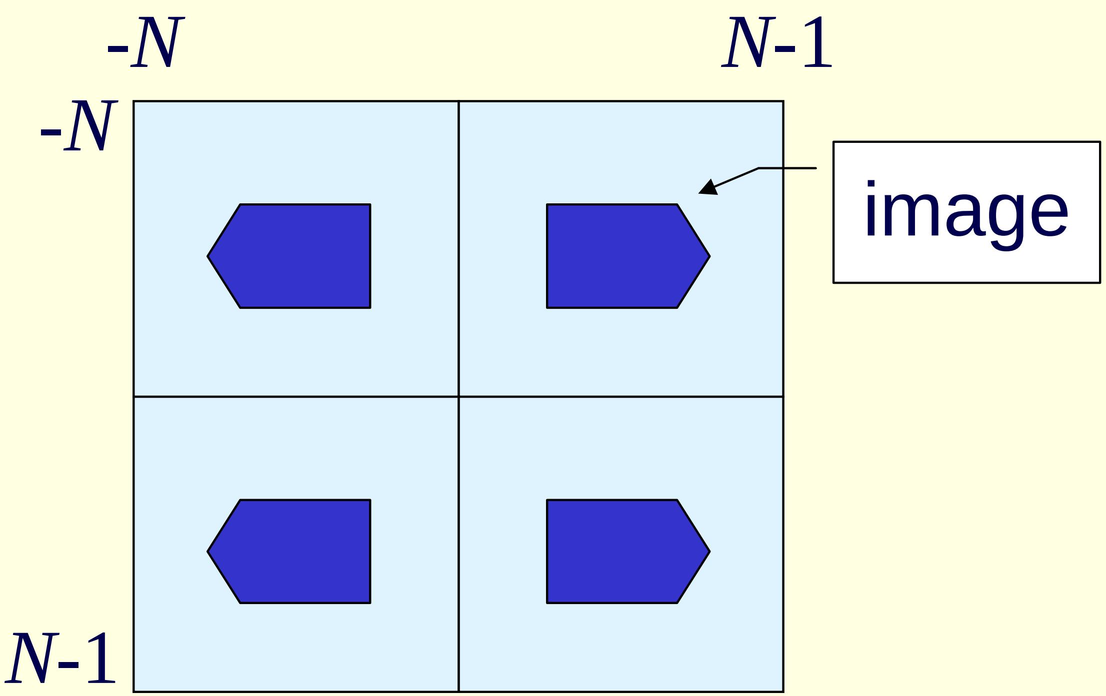


Discrete Cosine Transform (DCT)

$$F(u,v)=\frac{2}{N}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1} f(x,y)\cos\left[\frac{\pi u(2x+1)}{2N}\right]\cos\left[\frac{\pi v(2y+1)}{2N}\right]$$

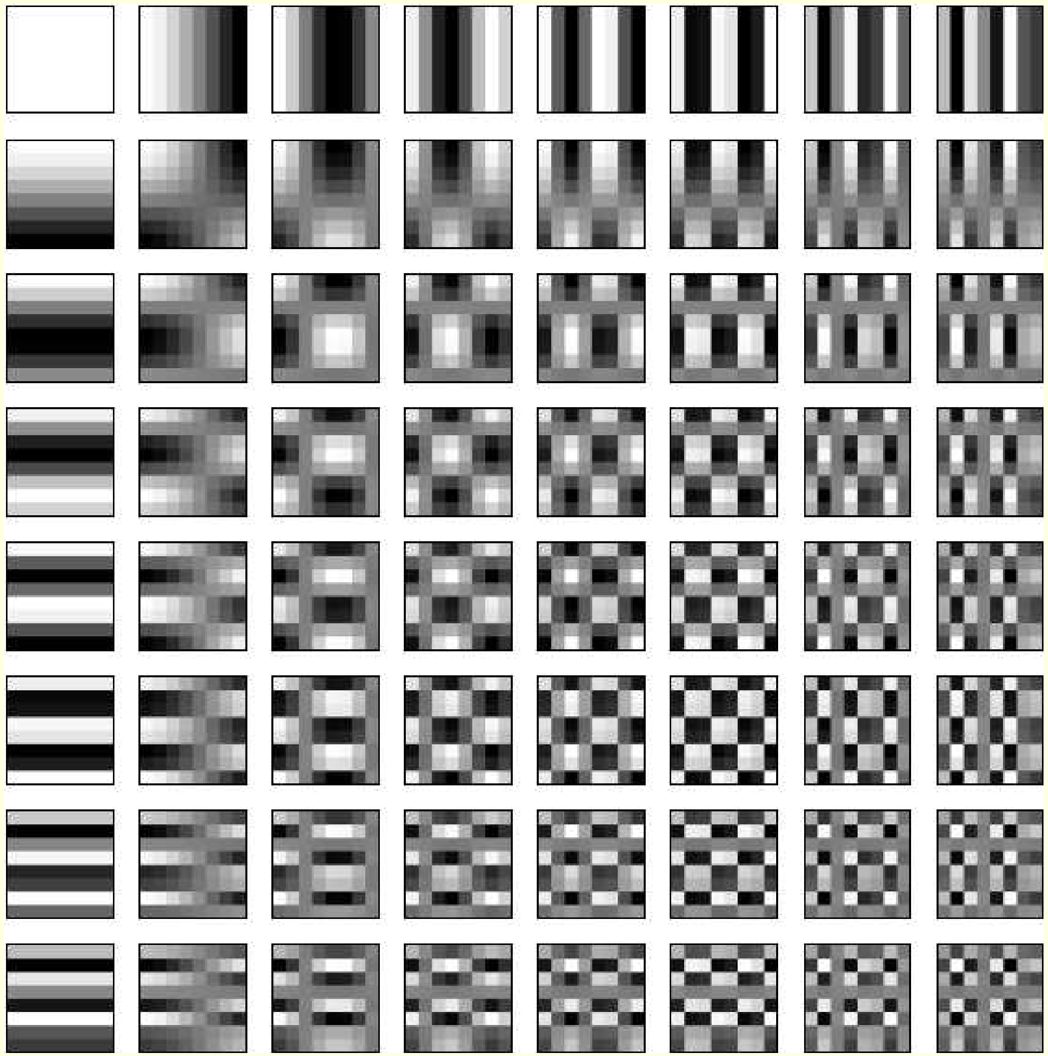
for: $u,v=1,2,\ldots,N-1$

Fourier spectrum of a real valued and symmetric function has real valued coeffcients, ie. only those associated with the cosine components of the Fourier series



DCT basis functions

DCT basis functions for 8x8 image blocks



Discrete Cosine Transform (DCT)

fast vanishing of the coefficients

(0,0)

‘autumn’ image

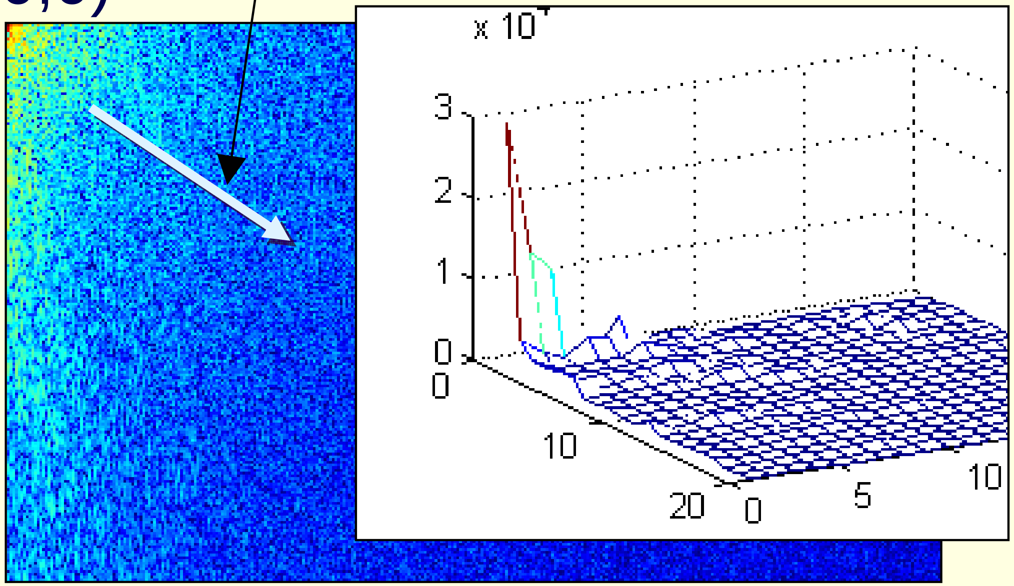
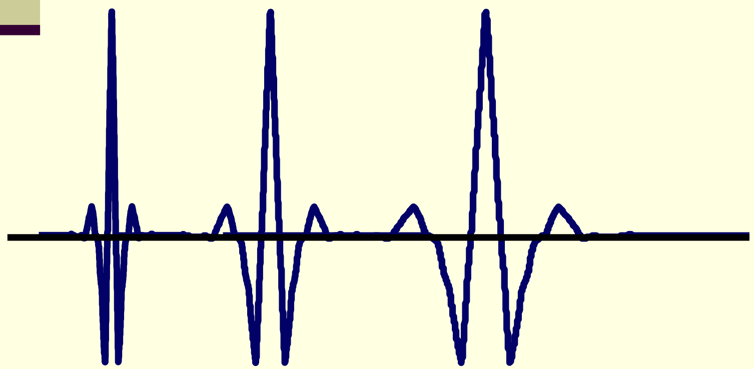


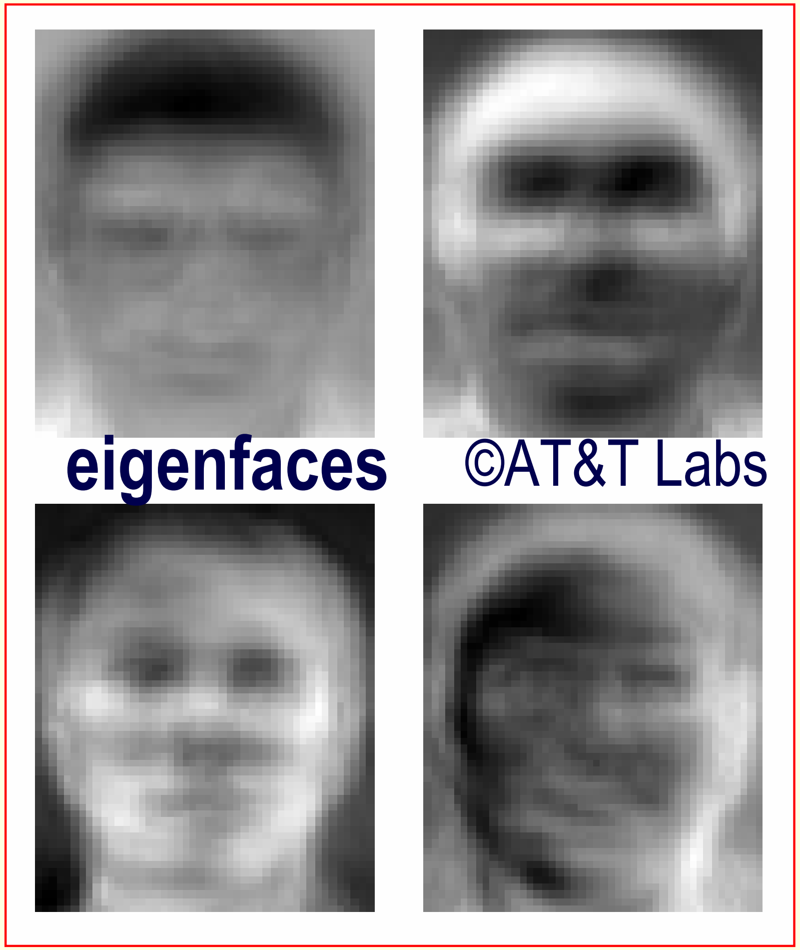
image cosine transform

The JPEG image compression standard is based on DCT

Other image transforms

- the **Karhunen-Loeve** transform - equivalent to the **PCA** (*Principal Component Analysis*)
- the wavelet transform is used in JPEG-2000 image coding standard





Other image transforms